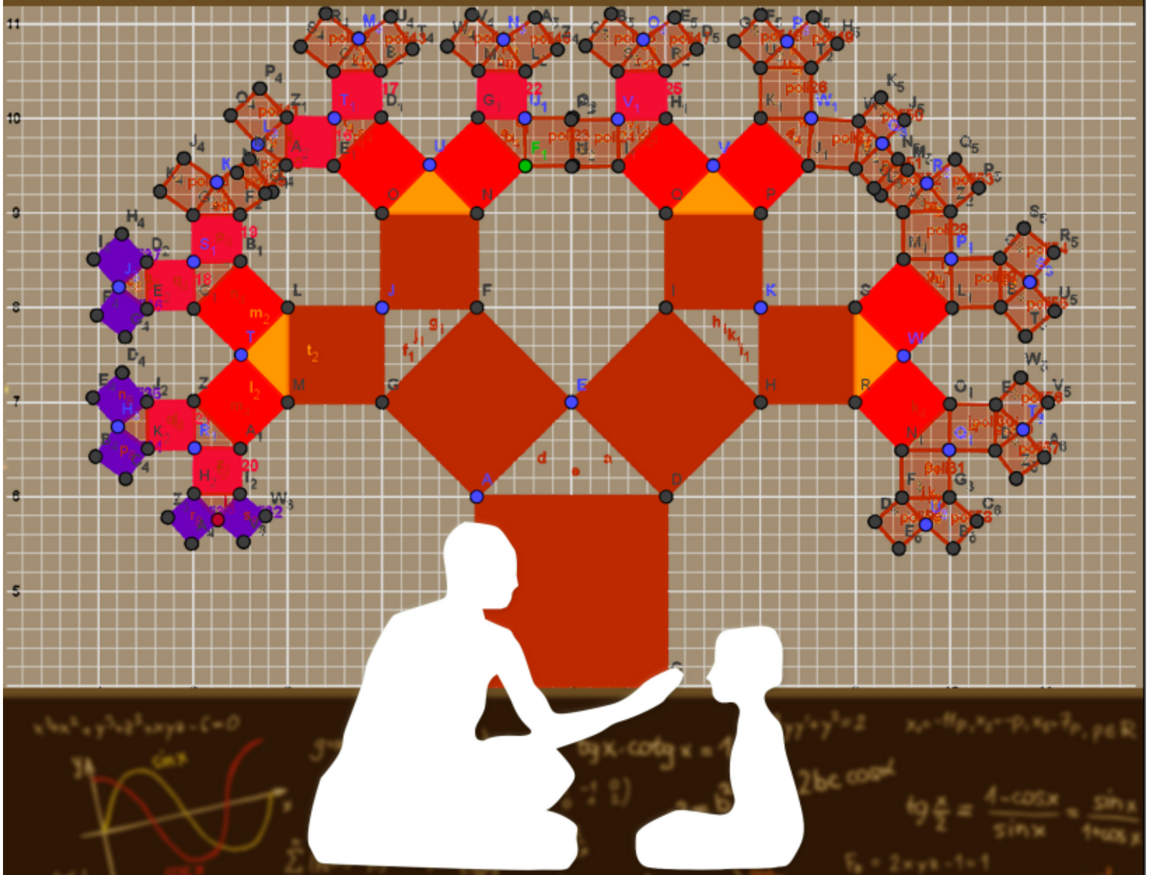
HIGHER SECONDARY

IT MATHS LAB

TEACHER'S HANDBOOK

FIRST YEAR



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Higher Secondary (First Year)

IT MATHS LAB

Teacher's Handbook

2019

State Council of Educational Research and Training
Poojappura PO
Thiruvananthapuram - 695012

Foreword

It is now a visual treat, inviting the students to explore the treasure troves of Mathematics. We have introduced IT Maths lab at the Higher Secondary level to maintain the continuity so that children could come up with a clear concept of the contents in the syllabus. The Teachers' Handbook to the IT Math lab manual explains each lab activity in clear and easy steps to facilitate the teacher while guiding the activities. There is room for innovation and creative intervention in each activity. We wish the teachers and students all success in these activities.

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Preface

The most important aim of mathematics education, as put forth by the National Curriculum Frame work 2005, is the mathematization of children's thoughts and ideas. KCF 2007 has observed that man is basically not only a recipient of knowledge but also a creator. The generation of any new knowledge comes naturally and thereby has a gladdening effect. It could be a fact long discovered, but if a child is able to wend his way to it on his own, he himself is the discoverer of the fact, as far as his thoughts go. It should instill in him no end of self-confidence and self-esteem. One of the ways in which this could be made to happen is to make mathematics learning as practical oriented as possible to help them see and feel what it is all about. At the primary level, mathematics is an exposition of concrete facts and their explanation. Hence there is more room for practical work at this level. When one moves on to the secondary and higher secondary levels the methodology of mathematics becomes abstract and technical and this limits the scope for introduction of practical oriented concepts. For example Calculus which deals with the study of rate of change is traditionally taught using methods that have, in sharp contrast, a stationary feel. We were able to bring in a move for the better with the introduction of geogebra, a software that keeps things on the move, to aid the study of calculus and similar topics. Even in classes where such changes, though on a small scale, could be effected, the students were just onlookers. It is with a view to over coming such constraints that the IT math Lab was conceived at the Higher Secondary level.

The lab activities have been explained in great detail in the manual so that a student who learns the basics of geogebra from Lab 0 can easily work out the lab activities on his own. When students are thus engaged in these activities, there is room for the teachers' intervention in each activity. The Teacher's Handbook has been prepared specifically to help the intervention. The Teacher's Handbook also gives models of questions to be given for lab evaluation. There are skills that students should acquire through each activity. This book will help the teachers in guiding students to achieve such skills and to discuss related matters with students. These areas are marked with an asterisk (*) and explained. For example in Lab-2 Shifting of graph: $f(x) + a$ we find 'Discuss the reason for vertical shift*'. It is explained at the bottom of the page under *, why the vertical shift takes place together with diagrams. This not only enhances the skills in lab activities but also helps class room teaching to a great extent.

All the labs in the manual need not be done in the lab. But, as the lab activities of each chapter are closely linked to class room activities, the lab manual could be used to construct applets which could be used to supplement class room work.

The capacity of children to adapt themselves quickly and easily to new technology could be useful in IT Math lab. The Teacher's Handbook has been so designed as to help teachers in the lab activities of students.

We earnestly hope that each teacher will be able to guide his students to fruition in the math lab activities, with all confidence. We wish the students and teachers a happy journey.

How To Use This Book

Some symbols are used in this book extensively for creating a new experience of reading. The meaning of the symbols is summarized below.

Symbol	Meaning
	**Student Activity** Indicates student involvement or action This could be the implementation of an instruction from the manual
	**Teacher's involvement** The teacher interfere's in the student's learning process The involvement of the teacher should be to guide the student to achieve a specific result
	**Findings** The findings of the student which has emerged out of the joint effort of both the teacher and student The results of this section would be the outcome of a typical active learning process
	**Discourse** Here aid is provided to the teacher on elaborating certain key concepts to the students Typical classroom teaching is involved in this section

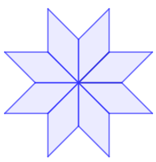
Contents

Basic Concepts

We explore different possibilities of using the software GeoGebra in our labs. GeoGebra is a software combining Geometry and Algebra and is very useful for visualizing various equations, functions etc. though not limited to these features. Even though most of the students are already familiar with GeoGebra from their high school classes we start from the basics. Different interfaces of GeoGebra and some basic tools and input commands - which are frequently used in the labs - are discussed here. Some Mathematical concepts are also discussed through different activities to familiarise the methods of an IT based Maths Lab. We also discuss about Activities, Additional Activities, Observation Book and the entries to be made in it (some models are given in the appendix) and the methods of saving the products of each lab (GeoGebra applets).

Activity 0.1 GeoGebra Interface

After giving a brief description about the interfaces and tools to the students, give some simple activities to familiarise important tools and their uses. Some examples are given below. Name of new tools are given in italics. The most important tool in GeoGebra is the Slider tool which is introduced in the next activity, Conic Section tools and 3D tools are discussed in detail in the relevant chapters, whose discussion is not attempted here.

Activity	**Tools**
Draw a triangle, find the length of its sides, perimeter, angles and area	*Polygon, Distance or Length, Angle, Area.* [1]
Draw a triangle of sides 5, 6 and 7 units	*Segment with Given length, Circle with Centre and Radius, Intersect*, Polygon [2]
Draw $\triangle ABC$ with $AB = 6$ units, $\angle A = 60°$ and $\angle B = 50°$	Segment with Given length, *Angle with Given size, Ray*, Intersect, Polygon. [3]
Draw a rectangle of sides 5 and 6 units	Segment with Given length, *Perpendicular Line*, Circle with Centre and Radius, Intersect, Polygon
Draw a parallelogram of sides 5 and 4 units	Segment with Given length, *Parallel Line*, Intersect, Polygon. [4]
Draw the pattern 	Segment with Given length, Angle, Parallel Line, Intersect, Polygon, *Reflect about Line.* [5]

1. Using Distance or Length tool click on each side to get length of the sides and click inside the triangle to get its perimeter. Using Area tool and Angle tool click inside the triangle to get the area and different angles of the triangle. While constructing the triangle make sure click the vertices in the anti-clockwise direction. Otherwise when we use the angle tool to get the angles of the triangle, it may show the exterior angles. We can draw a polygon either using Polygon Tool or using line segments. Discuss the difference between these two constructions. If we use line segments it will not be considered as polygon. We can't find its area or perimeter.

2. Discuss the methods of hiding the objects used for the construction. We can do it from the Algebra view by clicking on the bullet given on the left side of the object or by right clicking on the object and unchecking the item Show Object. Name of the object may be displayed in the graphics view. We can hide it by right clicking on the object and unchecking the item Show Label.

3. While using Angle with given size tool remember to select counter clockwise or clockwise as is the requirement.

4. This parallelogram is not unique.

5. Draw a Rhombus with one angle as 45°. Using Reflect about Line tool take its reflection on one of its sides. Repeat the process and complete the pattern.

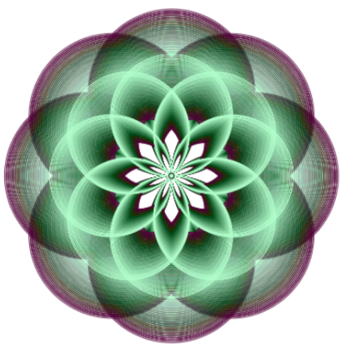
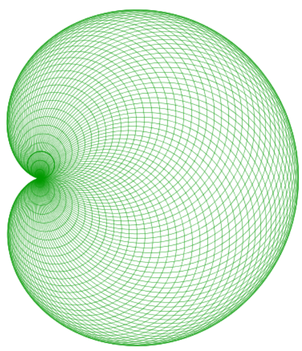
Activity 0.2 Graph of a Function

Aim of this activity is to introduce the Slider tool, Trace and Input Box. Students also get a clear idea about the graph of a function f as the set of points $(x, f(x))$.

Discuss the fact that the curve traced by the point $A(a, a^2)$ is the graph of the function $f(x) = x^2$. (a, a^3) gives the graph of $f(x) = x^3$ and so on

More examples may be given to get a practice on Slider tool and Trace. Some are given below.

Draw the following patterns

Construction of the patterns

Both patterns are constructed using circles.

Pattern 1

- ☛ Create a slider a with Min=2, Max=5 and increment 0.05
- ☛ Draw a circle of radius a, centered at a point A. Plot a point B on it.

- Plot another point B' on the circle so that $\angle BAB' = 45°$ (Use `Angle with given size tool` or `Rotate about a Point` tool). Repeat this until we get eight equidistant points on the circle.
- Draw circles centered at above points and passing through A.
- Select a suitable colour for above circles and `Trace on` them
- Hide the base circle, points and angles.
- Give animation of the slider.
- To get a dynamic colour to the pattern, do as follows
  - create three sliders r, g, and b with Min=0, Max=1 and different increments say 0.01, 0.02, and 0.015.
  - Select all circles (using `Move` tool right click and drag), right click inside the selection → Object properties → Advanced → Dynamic Colors → fill the columns of Red, Green and Blue by r, g and b.
  - Animate all sliders.

Pattern 2

- Draw a circle of convenient radius (say 3) with center at the point A and plot a point B on it.
- Create an angle slider α
- Plot another point B' on the circle so that $\angle BAB' = \alpha$
- Draw the circle centered at B' and passing through B.
- Select a suitable colour for above circle and `Trace on` it
- Hide the first circle, the points and angle.
- Animate the slider.

Activity 0.3 Standard Functions

Aim of this Activity is to familiarise the use of input box by drawing the graphs of some standard functions. More input commands are given in the appendix.

Ceiling function and Floor function are discussed. Graph of the Ceiling function is obtained by shifting the graph of the greatest integer function (Floor function) upwards by one unit. Notation of the Ceiling function is $\lceil x \rceil$ and that of the floor function is $\lfloor x \rfloor$ (we use the notation $[x]$). $\lceil x \rceil$ is defined as the least integer which is not less than x

Discuss the methods of finding domain and range of functions from their graphs.

Activity 0.4 Domain and Range

We discuss the domain and range of $f(x) = x^n$ when n is a natural number, and the behaviour of the graph as n increases.

- Domain of $f(x) = x^n$ is $\mathbb{R}$ for all values of n (natural number)
- Range is $\mathbb{R}$ if n is odd
- Range is $[0, \infty)$ if n is even
- As n increases, the portion of the graph in $(-1, 1)$ approaches more and more close to the x axis. This is because of the reason that if $x \in (-1, 1), x^n$ approaches to 0 as n increases to ∞.

Activity 0.A Polynomial Function

In this activity we discuss how the domain and range of a polynomial function related to its degree.

- Domain of all polynomial functions is $\mathbb{R}$
- Range of a polynomial function of odd degree is $\mathbb{R}$
- Range of a polynomial of even degree (n) is of the form $[a, \infty)$, if the coefficient of x^n is positive. If it is negative, range is of the form $(-\infty, a]$, where a is a real number

Activity 0.B Functions With Rational Powers

Here we discuss the nature of the function $f(x) = x^{\frac{1}{n}}$, where n is a natural number

- If n is even, graph of the function appears only on the positive side of the x axis.
- If n is odd, graph of the function appears on both sides of the x axis.
- If n is even, domain of the function is $[0, \infty)$, and range is $\mathbb{R}$.
- If n id odd, domain and range is $\mathbb{R}$
- If n is odd graph of $x^{\frac{1}{n}}$ is obtained by reflecting the graph of x^n on the line $y = x$
- If n is even, graph of $x^{\frac{1}{n}}$ is obtained by reflecting portion on the positive side of the x axis of the the graph of x^n on the line $y = x$.

Lab 1

Value of functions

This lab consists of three activities and an additional activity. All these activities help the student to get a clear graphical idea about functions and functional relationship between two variables.

Required Concepts

☛ Image of a real number under a function f
☛ Graph of a function f is the set of points $(x, f(x))$ for all x in the domain of f

Aim

Students construct a simple applet for which they use above concepts. Using the applet they establish, graphically, the relation between a real number and its image under the function f and use the applet to find images of some real numbers. Activities 1.1 and 1.2 helps them to concretise above concepts.
In Activity 1.3 with the help of a GeoGebra applet we compare a function with a machine which gives an output, according to the definition of the function, for a given input

Activity 1.1 Functions

In this activity we use the function $f(x) = x^2$

Drag the point A along the x axis and observe the movement of C	☛ What happens to C while A moves ? ☛ What is the relation between the x coordinate of A and y coordinate of C ? Why ?	☛ C moves along Y axis as A moves along X axis ☛ y coordinate of C is the square of x coordinate of A, because the function is $f(x) = x^2$
Problems	How can we use this applet to find the square of a number ?	Adjust the slider so that the x coordinate of A becomes the given number and find the corresponding y coordinate of C

Activity 1.2 Values of Functions

Set a suitable function and finds required values	Discuss different methods of doing the problem. For example, we can find the value of $3^{\frac{1}{3}}$ either by setting the function as $x^{\frac{1}{3}}$ and the value of the slider as 3 or by setting the function as 3^x and the value of the slider as $\frac{1}{3}$ (if exponential functions are already discussed)	Completes the table. [1]
Observe the movement of the point C when the point A comes closer and closer to the origin from both sides of origin	What happens to the value $f(a)$ as **a** approaches zero from right as well as from left ? The concept of limit and its relevance may be mentioned	When **a** approaches zero from the right side, $f(a)$ increases to infinity and when **a** approaches zero from the left, $f(a)$ decreases to negative infinity
Change the function to $f(x) = [x]$ and observe the movement of C according to A	Discuss the range of $f(x) = [x]$	☛ While A moves in between two integers, C stay on the least integer among them and jumps to the next integer as A moves to the next interval ☛ Range of f is the set of integers

1. The completed table is shown here

	$3^{\frac{1}{3}}$	$\sqrt{1.8}$	$2^{\frac{2}{3}}$	$\sqrt{\sqrt{5}}$	$(3.46)^{\frac{-3}{2}}$
Function	$x^{\frac{1}{3}}$	$\sqrt{x}$	$x^{\frac{2}{3}}$	$\sqrt{x}$	$x^{\frac{-3}{2}}$
Input(x)	3	1.8	2	$\sqrt{5}$	3.46
value of f(x)	1.4422	1.3416	1.5874	2.2361	0.1554

Activity 1.3 Function Machine

By giving the values as inputs,in the machine, obtain the outputs.	☛ Compare a function with an input output machine ☛ Discuss the domain of a function	Identifying the values for which the warning light of the machine turns red

Additional Activities

Activity 1.A Temperature Scales

This is an example of a life situation in which a functional relationship between two varying quantities are used and their comparison is done.Also this activity shows how mathematics is related to other subjects.

We may need to zoom out the graphics view to view the graph. Coordinates of the point of intersection of the graph with coordinate axes gives the answer to the questions given in the manual.

Lab 2

Shifting of Graphs

In this Lab there are four activities and three additional activities. All the activities deals with the graph of functions obtained by translation or reflection of graph of a given function.

Required concepts

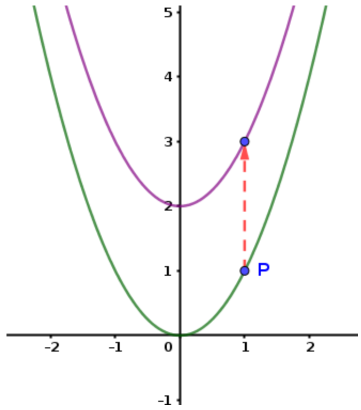
☛ Graph of a function

Aim
This Lab helps the students to imagine the graph of functions which are obtained by translation and reflection of the graphs of standard functions and hence to find their domain and range.
Activity 2.1 and 2.2 deals with shifting of graphs parallel to coordinate axes. Concept of family of curves is also discussed here.
Activity 2.3 and 2.4 deals with reflection of graphs on coordinate axes.
Translation of graphs, which are not parallel to coordinate axes are discussed in Additional activities 2.A and 2.B
Additional activity 2.C deals with the construction of family of curves using sequence command.

Activity 2.1 Shifting of graphs : $f(x) + a$

Observe the movement of the graph of $f(x) + a$ according to **a** for different functions	☛ Discuss the reason for vertical shift [1] ☛ Let the students imagine and draw rough sketch of the graphs of some functions such as x^2+3, x^2-2, $\|x\|-3$, $\|x\|+1$, x^3+2, $\frac{1}{x}+4$, $[x]+2$ etc. and check their answer using GeoGebra ☛ Discuss the domain and range of above functions. ☛ Discuss the concept of family of curves represented by the equations x^2+a, $\|x\|+a$ etc.	☛ Graph of $f(x)+a$ is obtained by shifting the graph of $f(x)$ by **a** units vertically upwards if **a** is positive and vertically downwards if **a** is negative. ☛ Finds the domain and range of functions from their graphs. Realise that vertical shift doesn't change the domain and may change the range.

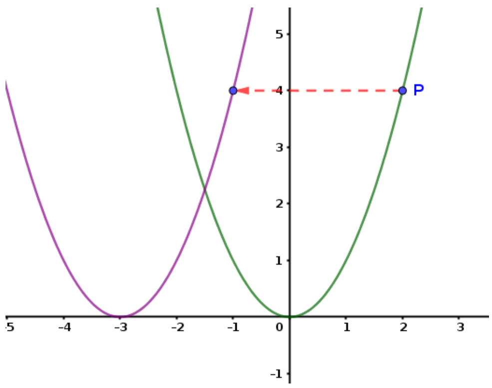
1. Consider the functions $f(x) = x^2$ and $g(x) = x^2+2$ ($g(x) = f(x)+2$). For a fixed x, say $x = 1$, $f(1) = 1$ and the corresponding point on the graph of f is P(1,1). $g(1) = f(1) + 2 = 3$ and the corresponding point on the graph of g is (1,3).That is, P is shifted vertically upwards by 2 units. Thus the graph of g is obtained by shifting each point on the graph of f, that is, the graph of f itself, vertically upwards by 2 units.



Activity 2.2 Shifting of graphs : $f(x+a)$

Observe the movement of the graph of $f(x+a)$ according to **a** for different functions	☛ Discuss the reason for horizontal shift [1] ☛ Let the students imagine and draw rough sketch of the graphs of some functions such as $(x+3)^2$, $(x-2)^2$, $\lvert x-3\rvert$, $\lvert x+1\rvert+2$, $1+(x+2)^3$, $\frac{1}{x+4}$, $[x-2]$, $[x-3]-1$ etc. and check their answer using GeoGebra ☛ Discuss the domain and range of above functions. ☛ Discuss the concept of family of curves represented by the equations $(x+a)^2$, $\lvert x+a\rvert$ etc.	☛ Graph of $f(x+a)$ is obtained by shifting the graph of $f(x)$ by **a** units parallel to x axis towards left if **a** is positive and towards right if **a** is negative. ☛ Finds the domain and range of functions from their graphs. Realise that horizontal shift doesn't change the range and may change the domain

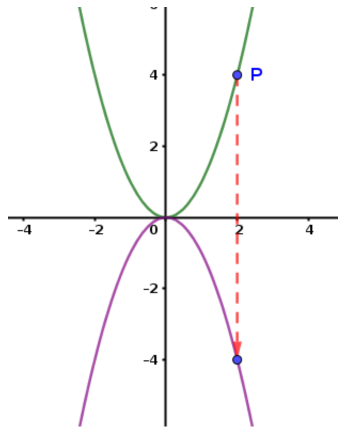
1. Consider the functions $f(x) = x^2$ and $g(x) = (x+3)^2$ (that is $g(x) = f(x+3)$). Since $2^2 = 4$, P(2,4) is a point on the graph of f. And since $(-1+3)^2 = 4$, $(-1, 4)$ is the corresponding point, that is the point with same y coordinate, on the graph of g. That is, P is shifted horizontally towards left by 3 units. Thus the graph of g is obtained by shifting each point on the graph of f, that is, the graph of f itself, horizontally towards left by 3 units.



Activity 2.3 Reflection of a Graph : $-f(x)$

Observe the graph of $-f(x)$ for different functions	☛ Discuss the reason for reflection [1] ☛ Let the students draw rough sketch of the graphs of some functions such as $-(x+3)^2$, $-(x-2)^2$, $-\|x-3\|$, $2-\|x+1\|$, $1-(x+2)^3$, $-\dfrac{1}{x+4}$, $-[x-2]$ etc. and check their answer using GeoGebra ☛ Discuss the domain and range of above functions.	☛ Graph of $-f(x)$ is obtained by reflecting the graph of $f(x)$ on x axis. ☛ Finds the domain and range of functions from their graphs. Realise that reflection on x axis doesn't change the domain and may change the range.

1. Consider the functions $f(x) = x^2$ and $g(x) = -x^2$ ($g(x) = -f(x)$). For a fixed x, say $x = 2$, $f(2) = 4$ and the corresponding point on the graph of f is P(2,4). $g(2) = -f(2) = -4$ and the corresponding point on the graph of g is (2,-4).That is, P is reflected on x axis. Thus the graph of g is obtained by reflecting each point on the graph of f, that is, the graph of f itself, on x axis.



Activity 2.4 Reflection of a Graph : $f(-x)$

Observe the graph of $f(-x)$ for different functions	☛ Discuss the reason for reflection [1] ☛ Let the students draw rough sketch of the graphs of some functions such as $(2-x)^2$, $\|3-x\|$, $\|x-1\|$, $1-(2-x)^3$, $-\dfrac{1}{4-x}$ etc. and check their answer using GeoGebra ☛ Discuss the domain and range of above functions. ☛ Discuss the concept of even and odd functions and their graphs [2]	☛ Graph of $f(-x)$ is obtained by reflecting the graph of $f(x)$ on y axis. ☛ Finds the domain and range of functions from their graphs. Realise that reflection on x axis doesn't change the range and may change the domain.

1. Consider the functions $f(x) = (x-1)^2$ and $g(x) = (-x-1)^2$ (that is $g(x) = f(-x)$). For $x = 3$, image of f is $(3-1)^2 = 4$ and the corresponding point on the graph of f is P(3,4). g attains this image at $x = -3$, $(g(-3) = (-(-3)-1)^2 = 4)$, and the corresponding point on the graph of g is (-3,4). That is, P is reflected on y axis. Thus the graph of g is obtained by reflecting each point on the graph of f, that is, the graph of f itself, on y axis.

   Note :- In the case of $f(x) = x^2$, students may not feel the reflection on y axis (because $(-x)^2 = x^2$) and in the case of $f(x) = x^3$ they may feel the reflection is on x axis (because $(-x)^3 = -x^3$). So it is better to consider the functions such as $f(x) = (x-1)^2$, $f(x) = x^3 + 1$ etc.

2. If f is even, $f(-x) = f(x)$, which shows that reflection on y axis doesn't change the the graph. So the graph of an even function is symmetric with respect to y axis. eg. x^2, $|x| + 2$ etc.

   If f is odd, $f(-x) = -f(x) \Rightarrow -f(-x) = f(x)$.
   Now, graph of $-f(-x)$ is obtained by reflecting the graph of $f(x)$ on x axis and then on y axis. In effect, the graph of f is reflected about the origin. So, for an odd function, $-f(-x) = f(x) \Rightarrow$ reflection about origin doesn't change the graph. So the graph of an odd function is symmetric about the origin. eg. x^3, $\frac{1}{x}$, etc.

   There are functions which are neither odd nor even. eg. $x^3 + 1$, $(x-2)^2$ etc.

Additional Activities

Activity 2.A Translations of Graphs:1

Observe the movement of the graph of $f(x-a)+a$ according to **a**	Discuss the reason for the shifting of curve parallel to the line $y = x$ [1]	Realise that the graph moves parallel to the line $y = x$
Changes the function to $f(x-a)-a$, $f(x-a)+2a)$ etc. and observe the movement of their graphs according to **a**.	Discuss the reason for the shifting of curve parallel to the lines $y = -x$, $y = 2x$ etc. [2]	Realise that the graph moves parallel to the lines $y = -x$, $y = 2x$ etc.

1. If **a** is positive, graph of $f(x-a)+a$ is obtained by shifting the graph of $f(x)$ by **a** units parallel to x axis towards right and then by **a** units parallel to y axis upwards. That is each point on the graph is shifted along a line having slope 1. So the entire graph is shifted parallel to the line $y=x$. (If **a** is negative, then also the shifting is parallel to the line $y=x$, but in the opposite direction)

2. If **a** is positive, graph of $f(x-a)-a$ is obtained by shifting the graph of $f(x)$ by **a** units parallel to x axis towards right and then by **a** units parallel to y axis downwards. That is each point on the graph is shifted along a line having slope -1. So the entire graph is shifted parallel to the line $y=-x$.

   If **a** is positive, graph of $f(x-a)+2a$ is obtained by shifting the graph of $f(x)$ by **a** units parallel to x axis towards right and then by **2a** units parallel to y axis upwards. That is each point on the graph is shifted along a line having slope 2. So the entire graph is shifted parallel to the line $y=2x$. (If **a** is negative, then also the shifting is parallel to the line $y=2x$, but in the opposite direction)

   The pattern given in the Manual is obtained by tracing the curves $(x-a)^2+a$ and $(x+a)^2+a$

Activity 2.B Translations of Graphs: 2

Observe the movement of the graph of $f(x+a)+b$ according to **a** and **b**	Discuss the reason for the shifting of curve parallel to a line having slope $-\frac{b}{a}$	Realise that the graph moves parallel to a line having slope $-\frac{b}{a}$

Activity 2.C Family of curves - using sequence command

Draw patterns using input commands	Discuss the the logic of input commands	☛ Draw different patterns ☛ Finds different input commands to draw their own patterns

Lab 3

Domain and Range

In this lab we have three activities and four additional activities. We discuss the domain and range of functions using their graphs, rational functions and piecewise defined functions. In additional activities some applications in Mathematics and Physics are discussed. Domain and range of relations are also given as an additional activity.

Required concepts

- Graph of a function, Domain and Range, Shifting of the graph

Aim

- Graph of a function can tell a lot of its properties such as domain, rage, maxima, minima, increasing/decreasing etc. Having a knowledge about the relation of the graph with its domain and range of some standard functions, students can explore it to more functions with the help of shifting reflection and scaling of graphs. In Activity 1 students get an opportunity to think about domain and range from graphs of functions and verify it using GeoGebra.
- In Activity 2 Rational functions and their behaviour at the points where they are not defined are discussed.
- Activity 3 deals with piecewise defined functions, their domain and range.
- Some practical problems are discussed in Activity 3.A, 3.B and 3.C which helps the students to relate Mathematics and Physics.
- Activity 3.D deals with the domain and range of relations graphically.

Activity 3.1 Domain and Range of Functions from their Graphs

Imagines the graph of the functions, their domain and range	Discuss the logical approach to the problems *	- Gets a clear idea about finding domain and range from graphs. - Finds the domain and range.

*

No:	Function	Discussion	Domain	Range
i	x^2+2	Vertical shift of x^2 by 2 units upwards	$\mathbb{R}$	$[2,\infty)$
ii	x^2-3	Vertical shift of x^2 by 3 units downwards	$\mathbb{R}$	$[-3,\infty)$
iii	$3-\|x\|$	read as $-\|x\|+3$. Reflection of $\|x\|$ on x axis and an upward shift of 3 units	$\mathbb{R}$	$(-\infty,3]$
iv	$(x+2)^2-1$	Left shift o x^2 by 2 units and down shift by 1	$\mathbb{R}$	$[-1,\infty)$
v	$x^2-6x+12$	Re write as $(x-3)^2+3$. Right shift and up shift of x^2 by 3 units	$\mathbb{R}$	$[3,\infty)$
vi	$\|x-2\|$	Right shift of $\|x\|$ by 2 units	$\mathbb{R}$	$[0,\infty)$
vii	$\|x-2\|+3$	Up shift of the previous graph by 3 units	$\mathbb{R}$	$[3,\infty)$
viii	$2x^2-8x+5$	Re write as $2(x-2)^2-3$. Right shift of x^2 by 2, multiplication by 2 do not change the domain and range (this is not the general case)[1]. Down shift by 3.	$\mathbb{R}$	$[-3,\infty)$
ix	$\frac{1}{2}[x]$	Since $[x]$ is defined for all real numbers $\frac{1}{2}[x]$ is also defined for all real numbers. Range of $[x]$ is $\mathbb{Z}$, so the range of $\frac{1}{2}[x]$ is the set of half of integers. Discuss the change in the shape of the graph[2]	$\mathbb{R}$	$\{x : x=\frac{n}{2}, n \in \mathbb{Z}\}$
x	$[\frac{x}{2}]$	Whatever may be in the bracket, we get its value as an integer and if we take $x=2n$, we get n as its image, for any natural number n. Discuss the change in the shape of the graph[3]	$\mathbb{R}$	$\mathbb{Z}$
xi	$x-[x]$	Defined for all values of x. For any x its integer part will be get cancelled. So $f(x) \in [0,1)$. Discuss the graph[4]	$\mathbb{R}$	$[0,1)$
xii	$3-x^2$	Read as $-x^2+3$. Reflection of x^2 on x axis and an up shift by 3	$\mathbb{R}$	$(-\infty,3]$
xiii	$\sqrt{x-2}$	Domain: $x-2 \geqslant 0 \Rightarrow x \geqslant 2$. Range : minimum value of the function is 0 and it increases to ∞ as x increases. Discuss the graph[5]	$[2,\infty)$	$[0,\infty)$
xiv	$\sqrt{4-x}$	Domain: $4-x \geqslant 0 \Rightarrow x \leqslant 4$. Range : minimum value of the function is 0 and it increases to ∞ as x decreases. Discuss the graph[6]	$(-\infty,4]$	$[0,\infty)$
xv	$\dfrac{1}{x-2}$	Right shift of $\frac{1}{x}$ by 2 units	$\mathbb{R}-\{2\}$	$\mathbb{R}-\{0\}$
xvi	$\sqrt{x^2-4}$	Domain : $x^2 \geqslant 4 \Rightarrow x \leqslant -2$ or $x \geqslant 2$ Range : minimum value of the function is 0 and it increases as x increases to $+\infty$ or decreases to $-\infty$. Discuss the graph[7]	$\mathbb{R}-(-2,2)$	$[0,\infty)$
xvii	$\sqrt{9-x^2}$	Domain : $x^2 \leqslant 9 \Rightarrow -3 \leqslant x \leqslant 3$ Range : minimum value 0 and maximum value 3. Discuss the graph[8]	$[-3,3]$	$[0,3]$
xviii	$\frac{1}{x^2-9}$	Discuss Domain and range[9]	$\mathbb{R}-\{-3,3\}$	$\mathbb{R}-(-\frac{1}{9},0]$
xix	$\frac{x^2}{x^2+1}$	Discuss Domain and range[10]	$\mathbb{R}$	$[0,1)$

1. In this case the domain of the function f is $\mathbb{R}$ so the domain of $2f$ is also $\mathbb{R}$. The range of f is $[0, \infty)$ so the range of $2f$ is also $[0, \infty)$ (in both cases the minimum value of the functions is 0 and their value increases to ∞ as x increases). Consider the functions $f(x) = x^2 + 2$. Its domain is $\mathbb{R}$ and so the domain of $2f$ is also $\mathbb{R}$. The range of f is $[2, \infty)$ so the range of $2f$ is $[4, \infty)$ (in this case the minimum value of f is 2 hence that of $2f$ is 4. Both the functions increases to ∞.)

2. The graph is vertically compressed to half. That is the space between the steps is reduced to half of the original.

3. The graph is horizontally elongated to twice. That is the length of each step is doubled.

4. In such examples (if functions are discontinuous) it is not easy, by mere observation of the graph, to say whether some points are on the graph or not. For example we may doubt whether (2,0) or (2,1) are on the graph of the above given function. In such situations either we can go back to the definition of the function or using an input command to get a clarity. Input $f(2)$, we can see its value in the algebra window as 0. which means $(2, 0)$ is on the graph hence $(2, 1)$ is not. If we input $(2, f(2)$ we get the point $(2, 0)$ on the graph. Suppose 2 is not in the domain of a function f, while we give the input, algebra window will show it undefined.

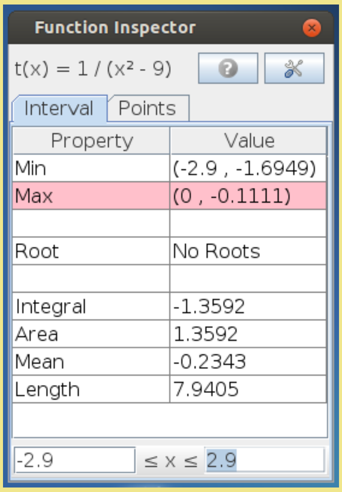
5. We can come back to this discussion after the chapter Conic Sections. $y = \sqrt{x-2} \Rightarrow y^2 = x - 2$ which is a shifting of the parabola $y^2 = x$. So, starting from $y = \sqrt{x}$ which represents the upper part of the parabola $y^2 = x$, we get the graph of the function $y = \sqrt{x-2}$ by shifting above graph by two units right.

6. $y = \sqrt{4-x} \Rightarrow y^2 = 4 - x$. So we start from the parabola $y^2 = -x$. The function $y = \sqrt{-x}$ represents the upper half of above parabola. Hence the graph of $y = \sqrt{4-x} = \sqrt{-(x-4)}$ is obtained by shifting above graph by 4 units right.

7. $y = \sqrt{x^2-4} \Rightarrow \frac{x^2}{4} - \frac{y^2}{4} = 1$ is a Hyperbola with vertices $(\pm 2, 0)$. So the given function represents the upper half of this Hyperbola.

8. $y = 9 - x^2 \Rightarrow x^2 + y^2 = 9$ is the circle of radius 3 centered at the origin. So the given function represents the upper half of this circle.

9.
   - Domain : $\mathbb{R} - \{-3, 3\}$
   - Range : $x^2 - 9$ takes all positive values so $\frac{1}{x^2-9}$ also take all positive values (as in the case of $\frac{1}{x}$). On the negative side (that is when $x^2 - 9$ is negative), $x^2 - 9$ takes the values from -9 to 0 . So $\frac{1}{x^2-9}$ takes the values from $-\infty$ to $-\frac{1}{9}$. Hence the range is $(-\infty, -\frac{1}{9}] \cup (0, \infty)$. That is $\mathbb{R} - (-\frac{1}{9}, 0]$
   - It may feel difficult to identify the maximum value of the function on the negative side by mere observation. Here we can use the Function Inspector tool. Using the tool, click on the graph, a window will appear. Adjust the value of x between any two numbers in $(-3, 3)$ say as $-2 \leqslant x \leqslant 2$. We get the maximum and minimum values of the function in that interval.



1. ☛ Domain : R
   ☛ Range : Since the numerator and denominator are positive, value of the function is positive. It takes the value 0, when $x = 0$. Since the denominator is greater than the numerator, value of the function is less than 1. Now the question remains is that does it take all values in $[0, 1)$. We can consider the function as $1 - \frac{1}{x^2+1}$. Since $\frac{1}{x^2+1}$ takes all values in $(0, 1]$, $1 - \frac{1}{x^2+1}$ takes all values in $[0, 1)$.
   ☛ If it feel difficult to identify the maximum value of the function on the negative side by mere observation of the graph, use Function Inspector tool. Using the tool click on the graph. A window will appear. Adjust value of x between any two numbers in $(-3, 3)$ say as $-2.9 \leqslant x \leqslant 2.9$. We can see the maximum value of the function in $(-2.9, 2.9)$.

Activity 3.2 Rational Functions

Creates the applet, Observes the movement of the point along the graphs.	Discuss the difference between the functions[1]	☛ Identifies the difference between the functions.

1. Even though the graphs of the functions seems to be the same they are not actually like that. Since the first function is not defined at 2, (2,4) is not a point on the first graph. That is why the point disappears at $x = 2$. But (2,4) is on the second graph.

Activity 3.3 Piecewise Functions

Draws the graphs of the functions, observes the graphs to find the domain and range of the functions.	Help the students to identify the domain and range. *	Finds the domain and range.
Observes the given graphs	Discuss the function representing the graphs **	Identifies the functions

*

No:	Function	Domain	Range
1	$f(x) = \begin{cases} x^2 & \text{if } x \leq 2 \\ 2x+1 & \text{if } x > 2 \end{cases}$	$\mathbb{R}$	$[0, \infty)$
2	$f(x) = \begin{cases} x^3 & \text{if } x \leq 0 \\ x^2+1 & \text{if } x > 0 \end{cases}$	$\mathbb{R}$	$\mathbb{R} - (0, 1]$ [1]
3	$f(x) = \begin{cases} x^2+2 & \text{if } x < 0 \\ -x^2-2 & \text{if } x > 0 \end{cases}$ [2]	$\mathbb{R} - \{0\}$	$\mathbb{R} - [-2, 2]$
4	$f(x) = x^2$ if $x \in [-2, 1]$	$[-2, 1]$	$[0, 4]$
5	$f(x) = x^3$ if $x \in [-2, 2]$	$[-2, 2]$	$[-8, 8]$
6	$f(x) = \frac{1}{x}$ if $x \in [-1, 2]$	$[-1, 2] - \{0\}$	$\mathbb{R} - (-1, \frac{1}{2})$

1. It is not possible to say whether the points $(0, 0)$ or $(0, 1)$ are on the graph, by mere observation. From the definition of the function it is clear that $f(0) = 0^3 = 0$. Hence $(0, 0)$ is the point on the graph and $(0, 1)$ is not. We can use input commands to see it from the graph itself. Input $f(0)$ shows its value on the algebra view, or the input $(0, f(0))$ shows the point on the graphics view.
2. Input command for this function is $If(x < 0, x^2 + 2, x > 0, -x^2 - 2)$.

**

	Function	Input Command
a	$f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ [x] & \text{if } x > 1 \end{cases}$	`If(x<=1,x^2,floor(x))`
b	$f(x) = \begin{cases} [-x] & \text{if } x \leq -1 \\ x^2 & \text{if } -1 < x < 1 \\ [x] & \text{if } x \geq 1 \end{cases}$	`If(x<=-1,floor(-x),-1<x<1,x^2,floor(x))`
c	$f(x) = \begin{cases} (x+2)^2 & \text{if } x \leq -1 \\ x^2 & \text{if } -1 < x < 1 \\ (x-2)^2 & \text{if } x \geq 1 \end{cases}$ [1]	`If(x<=-1,(x+2)^2,-1<x<1, x^2,(x-2)^2)`
d	$f(x) = \begin{cases} \lvert x+4\rvert - 1 & \text{if } x \leq -2 \\ \lvert x\rvert - 1 & \text{if } -2 < x < 2 \\ \lvert x-4\rvert - 1 & \text{if } x \geq 2 \end{cases}$ [2]	`if(x<=-2,abs(x+4)-1,-2<x<2,abs(x)-1,abs(x-4)-1)`

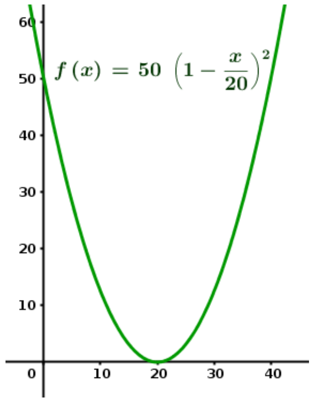
- We may restrict the function in $[-3, 3]$ as seen in the given figure. That is
$$f(x) = \begin{cases} (x+2)^2 & \text{if } -3 \leq x \leq -1 \\ x^2 & \text{if } -1 < x < 1 \\ (x-2)^2 & \text{if } 1 \leq x \leq 3 \end{cases}$$
- The corresponding input command is
`If(-3<=x<=-1,(x+2)^2,-1<x<1, x^2,1<=x<=3,(x-2)^2)`
- If we want a continuation of the picture we can use the Sequence command as follows. Create an integer slider m with min=0. Input the command
`Sequence(If(2n - 1 ≤ x ≤ 2n + 1, (x - 2n)²), n, -m, m, 1)`
As m increases we get a continuation of the picture
- The logic behind above command is that the function is defined as $(x - 2n)^2$ in the the interval $[2n - 1, 2n + 1]$. Using Sequence command we join such functions defined in different intervals. Note that the picture doesn't represent a single function, but a sequence of functions.
- We can restrict the function in $[-6, 6]$ using the input command
`if(-6<=x<=-2,abs(x+4)-1,-2<x<2,abs(x)-1,2<=x<=6,abs(x-4)-1)`
- The input command `Sequence(If(4n-2<=x<=4n+2, abs(x-4n)-1),n,-m,m)` gives a continuation of the picture

Additional Activities

Activity 3.A Leaking Tank

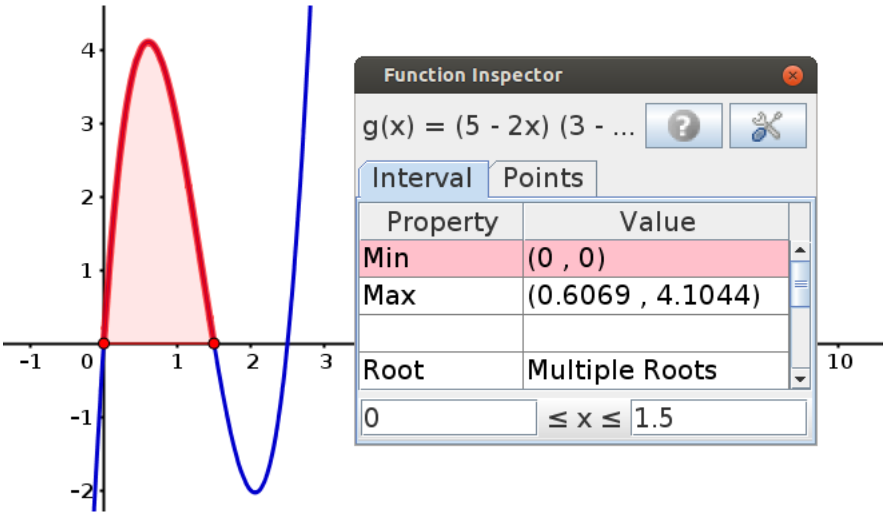
Draw the graph of the function $f(x) = 50(1 - \frac{x}{20})^2$. $f(0), f(5), ...$ gives the corresponding volume of water in the tank at time 0, 5, ...

Domain of the function $f(x) = 50(1 - \frac{x}{20})^2$ is the set of real numbers $\mathbb{R}$ and its range is $[0, \infty)$. But it is not the case in the given situation. The tank becomes empty in 20 minutes. So after 20, the volume of water in the tank is 0. But the function f increases from 0 after 20 (it is clear from the graph). So the domain of the function $V(t)$ is $[0, 20]$. The range is the volume of water in the tank with in the time interval 0 to 20 minutes. So the range is $[0, 50]$.



Activity 3.B The Volume of a Box

Volume of the box is given by the function $v(x) = (5 - 2x)(3 - 2x)x$. Since x represents the length of the square cut away from the cardboard, its minimum value is 0 and maximum value is $\frac{1}{2}$ (either we can exclude these numbers, because no box is there at these values or we can include them by considering boxes of 0 volume). Hence we can take the domain as $[0, \frac{3}{2}]$ or $(0, \frac{3}{2})$.
Range of the function is obtained from the graph. We can use the 'Function Inspector ' tool to find the maximum volume and the corresponding value of x.



Click on the graph with the 'Function Inspector' tool and set the value of x in $[0, 1.5]$. The maximum volume is obtained is 4.1044 at $x = 0.6069$ (both are approximate values).
The range of the function can be taken as $[0, 4.1044]$ or $(0, 4.1044]$.

Activity 3.C Some Familiar Graphs from Physics

All these graphs are related to motion on a straight line. Students are familiar with this kid of graphs prom their Physics classes.

First graph represents displacement time graph of a ball thrown vertically upwards.

Second graph represents displacement time graph of a bouncing ball.

Third graph represents the velocity time graph of a bouncing ball.

Fourth graph : descriptions on the axes are not given. If we take the time along x axis and the displacement along y axis, this may be considered as the displacement time graph of a car moving along a straight road. It moves for some time, takes rest and again moves. Instead of displacement, if we take velocity along y axis, we can describe its motion as, moves with an acceleration for some time, keeps a constant velocity and again moves with acceleration.

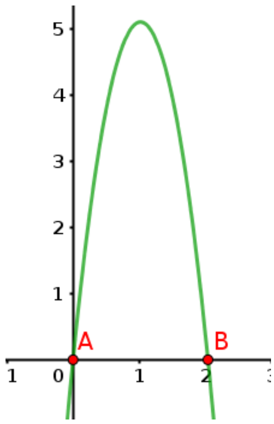
Construction of the graphs

Graph 1

Suppose the ball is thrown with an initial velocity of 10 m/s. Then its equation of motion is $h(t) = 10t - 4.9t^2$. We can draw its graph using the input command `h(x)=10x-4.9x^2`. We can restrict its domain in to its time of motion as follows.
Plot the points of intersections A and B of the curve with the x axis. Define the function $f(x) = h(x)$ in between the x coordinates of A and B using the command

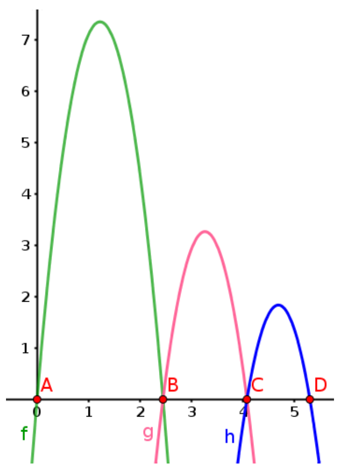
```
f(x)=if(x(A)<=x<=x(B),h).
```

Create a slider t with min=0 and max=x(B). Plot the point $(t, f(t))$ and animate t. We can see that above point moves along x axis as a ball ball thrown vertically upwards.



Graph 2

Initial velocity of the ball decreases on each bounce. Suppose the velocities are 12 , 8, 6, ... m/s. Graphically each to and fro motion is represented by a parabola. Second parabola starts where the first one ends and the third starts where the second one ends and so on. Graph of the first to and fro motion is obtained by the input `f(x)=12x-4.9x^2`. Plot the points of intersections A and B of the graph with x axis. Graph of the second to and fro motion is obtained by shifting the graph of the function $8x - 4.9x^2$ so that it starts at B. Use the input command `g(x)=8(x-x(B))-4.9(x-x(B))^2`. Plot the point of intersection C of this graph with x axis. Use the command `h(x)=6(x-x(C))-4.9(x-x(C))^2` to get the third portion.Now we can make it to a single graph by the command `if(x(A)≤x<x(B),f,x(B)≤x<x(C),g,x(C)≤x<x(D),h)`



Graph 3

This graph is obtained by taking the derivative of the function representing Graph 2.

Graph 4

This graph is a combination of the parabolas $y = -(x-2)^2 + 4$, $y = (x-4)^2 + 4$ and the straight line $y = 4$. It is obtained by the command

```
if(0≤x<2,-(x-2)^2+4,2<x<4,4,4≤x<6,(x-4)^2+4).
```

Activity 3.D Domain and Range of Relations

We can identify the domain and range from the regions represented by the relations.

Relation	Domain	Range
R_1	$[-2, 2]$	$[-2, 2]$
R_2	$[-2, 2]$	$[-2, 2]$
R_3	R	R
R_4	R	R
R_5	R	$[2, \infty)$

Lab 4

Trigonometric Functions

This lab consists of three activities and two additional activities. By these activities,students get a geometrical concept behind the definition of all the 6 trigonometric functions and their graphs. We also discuss the behaviour of each trigonometric function in different quadrants.

Required concepts

- Concept of circular functions, that is, the coordinate of a point rotates from $(1, 0)$ along the unit circle centered at the origin making an angle x radian at the centre can be taken as $(\cos x, \sin x)$
- Definition of other trigonometric functions on the basis of $\sin(x)$ and $cos(x)$
- Graph of the function f is a collection of points of the form (a, f (a)) for all values of a in its domain

Aim

- Students construct an applet in which they can rotate a point along a unit circle by a required angle. They find the values of trigonometric functions for given numbers. They use the concept of trigonometric functions for the construction of the applet.
- Using the applet, discuss the domain and range of the trigonometric functions
- With the help of the applet, plot the graphs of trigonometric functions (without using direct input command)
- Discuss the nature of trigonometric functions, such as Positive, Negative, Increasing, Decreasing etc. in different quadrants.

Activity 4.1 Values of Trigonometric Functions

☛ Creates the applet ☛ Observes the coordinates of P ☛ Finds the values of $\sin x$ and $\cos x$ for given values of x, using the applet.	☛ Recalls the definition of $\sin x$ and $\cos x$ ☛ Discuss the domain and range of $\sin x$ and $\cos x$ [1] ☛ What happens to the value of the functions when P reaches on the coordinate axes? ☛ Extend above discussion to find the values of x for which $\sin x$ and $\cos x$ takes the values $0, 1$ and -1	☛ Finds the domain and range of $\sin x$ and $\cos x$ ☛ Finds the values of $sinx$ and $\cos x$ for the given values of x and completes the table ☛ Identifies the values of x for which $\sin x$ and $\cos x$ takes the values $0, 1$ and -1

1. We may treat this as an input-output mechanism. We can input a value for the slider **a** to get the corresponding output, the values of $\cos a$ and $\sin a$, as the coordinates of P. Set of possible values of input, that is, the possible values of **a** is the domain of the functions. Even though we have created the slider with Min$= -10$ and Max$= 10$ we can edit them to any real number. So the domain of $\sin x$ and $\cos x$ is the set of real numbers. And the range is $[-1, 1]$

   Note :- While analysing a graph of a function, we take the domain as the set of x coordinates and range as the set of y coordinates. Don't let the students to confuse this with the above discussion.

x	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{6}$	$\frac{\pi}{2}$	.3	.6	2	-1.5	-3.1	7.5
$\sin x$	0.87	0.71	0.5	1	0.3	0.56	0.91	-0.997	-0.042	0.938
$\cos x$	0.5	0.71	0.87	0	0.96	0.83	-0.42	0.07	-0.99	0.347

Activity 4.2 Graphs of Trigonometric Functions - 1

Observes the path of B	☛ Recalls the concept that graph of the function f is the set of points $(x, f(x))$ ☛ $(a, y(P))$ represents the point $(a, \sin a)$, hence its path gives the graph of $\sin x$	Realise that the path of B is the graph of $\sin x$

Activity 4.3 Graphs of Trigonometric Functions - 2

Observes the path of B for different cases	☛ $(a, x(P))$ represents the point $(a, \cos a)$, hence its path gives the graph of $\cos x$ ☛ Recalls the definitions of other trigonometric functions. ☛ Since $y(P)$ represents $\sin a$, $\frac{1}{y(P)}$ represents cosec a. So the path of B$(a, \frac{1}{y(P)})$ gives the graph of cosec x and so on. ☛ Discuss the domain and range of trigonometric functions.	☛ Identifies the graphs ☛ Finds the definition of B to get the graphs of $\sec x$ and $\cot x$ ☛ Finds the domain and range of the trigonometric functions ☛ Observes the behaviour of the trigonometric functions in different quadrants and completes the table

Activity 4.A $k\sin(x)$

☛ Creates the applet ☛ Observes the coordinates of P ☛ Plots the functions $k\sin x$ and $k\cos x$	☛ Coordinates of points on a circle of radius k is $(k\cos x, k\sin x)$. (Idea of similar triangles may be used to establish this) ☛ Domain is $\mathbb{R}$ and range is $[-k, k]$ - discuss the reason.	☛ Realise that the coordinates of P is $(k\cos x, k\sin x)$ ☛ Finds the domain and range of the functions $k\sin x$ and $k\cos x$

Activity 4.B $k\sin(2x)$

☛ Creates the applet ☛ Observes the coordinates of P ☛ Plots the functions $k\sin 2x$ and $k\cos 2x$	☛ For any value of **a**, rotation of P is **2a**. Hence the coordinates of P is $(k\cos 2a, k\sin 2a)$. Hence its x coordinate represents the function $k\cos 2x$ and y coordinate represents the function $k\sin 2x$. ☛ Create a slider **b** and edit the rotation of P as **ab** to create the applet to describe the functions $k\sin ax$ and $k\cos ax$	☛ Realise that if $a = x$, the coordinates P is $(k\cos 2x, k\sin 2x)$. ☛ Creates the required applet

Lab 5

Trigonometric Identities

In this lab we derive some trigonometric identities geometrically and discuss their use in some physical situations through three activities and two additional activities.

Required Concepts

- Trigonometric functions are defined by means of coordinates of a point on the unit circle centred at origin
- Concept of congruent triangles

Aim

- Students construct an applet by which they derive the relation between $\sin(\frac{n\pi}{2} \pm x)$, $\cos(\frac{n\pi}{2} \pm x)$, $\tan(\frac{n\pi}{2} \pm x)$ with $\sin x$, $\cos x$, $\tan x$ for any natural number **n**. In Activities 5.1 and 5.2 they observe the results numerically and in Activity 5.3 they derive the results in general using the idea of congruent triangles.
- In Activity 5.A, students derive the result $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$.
- I Activity 5.B we discuss Simple harmonic Motion (SHM)

In the following activities, even though the identities are true for all values of x, we take $0 \leq x \leq \frac{\pi}{2}$ for the convenience of discussion

Activity 5.1 $\sin(\frac{n\pi}{2} + x)$

• Constructs the applet • Observes the coordinates of the points P and Q and finds the relation between them.	• Discuss how the function changes for even and odd values of n [1] • Discuss the method of finding the sign of the resultant function [2] • Use the definition $\tan x = \frac{\sin x}{\cos x}$ to derive the result of tan	• Completes the table • Establishes the relation between $\sin(\frac{n\pi}{2} + x)$, $\cos(\frac{n\pi}{2} + x)$, $\sin x$ and $\cos x$ for different values of n • Establishes the relation between $\tan(\frac{n\pi}{2} + x)$, $\tan x$ and $\cot x$ for different values of n

1. If n is even then the function is unaltered (the reason may be discussed in Activity 5.3). In this case we can write $\frac{n\pi}{2}$ in the form $n\pi$. So we can write the identities as follows
   $\sin(n\pi \pm x) = \pm \sin x$, $\cos(n\pi \pm x) = \pm \cos x$ and $\tan(n\pi \pm x) = \pm \tan x$

   If n is odd then the function changes as follows.
   $\sin(\frac{n\pi}{2} \pm x) = \pm \cos x$, $\cos(\frac{n\pi}{2} \pm x) = \pm \sin x$ and $\tan(\frac{n\pi}{2} \pm x) = \pm \cot x$

2. Sign of the expression on the right side of the equation is same as the sign of the expression on the right side which is determined by the quadrant in which the point Q lies after the rotation $\frac{n\pi}{2}$.
   For example
   $\sin(3\pi - x) = \sin x$, because $3\pi - x$ lies in the second quadrant where value of sine is positive
   $\cos(\frac{\pi}{2} + x) = -\sin x$ because $\frac{\pi}{2} + x$ lies in the second quadrant where value of cosine is negative

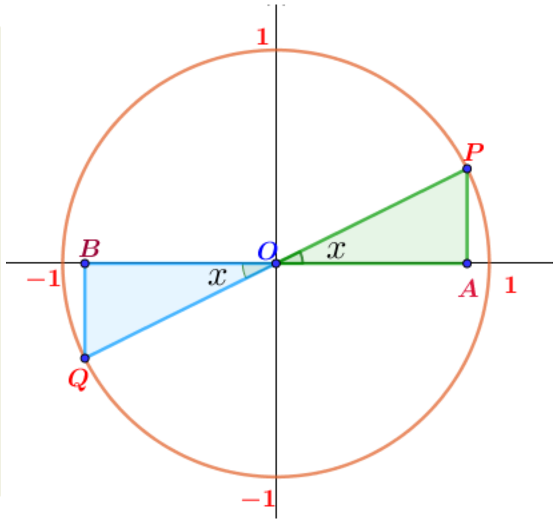
Activity 5.2 $\sin(\frac{n\pi}{2} - x)$

Observes the co-ordinates of the points P and Q and finds the relation between them.	☛ Discuss how the function changes for even and odd values of n ☛ Discuss the method of finding the sign of the resultant function	☛ Completes the table ☛ Establish the relation between $\sin(\frac{n\pi}{2} - x)$, $\cos(\frac{n\pi}{2} - x)$, $\sin x$ and $\cos x$ for different values of n ☛ Establishes the relation between $\tan(\frac{n\pi}{2} - x)$, $\tan x$ and $\cot x$ for different values of n

Activity 5.3 Geometrical Proof

Using the given applet proves the identities	☛ Use the concept of congruent triangles to get the results. [1] ☛ Discuss the reason that the functions are unaltered when n is even and changes when n is odd. [2] ☛ Discuss the relevance of such identities. [3]	Gets a geometrical picture of the identities

1. Rotation of Q is $\pi + x$
   Coordinates of P are $(\cos x, \sin x)$ and that of Q are $(\cos(\pi+x), \sin(\pi+x))$
   $\triangle OAP$ $\triangle OBQ$ are congruent
   So OA = OB and OP = OQ
   Since Q lies in the third quadrant its x and y coordinates are negative. So we get the results
   $\sin(\pi+x) = -\sin x$ and $\cos(\pi+x) = -\cos x$
   Similarly we can derive the other results.



2. When n is even, and if $x = 0$ both the points P and Q lie on the x axis. If we gradually increase the value of x we can see that both the points deviates from the x axis by the same distance. So that their y coordinates are same (except in sign - may be). So we get the result $\sin(n\pi \pm x) = \pm \sin x$. Congruency of the triangles gives the other result $\cos(n\pi \pm x) = \pm \cos x$.
   When n is odd, and if $x = 0$ P lies on x axis and Q lies on the y axis. If we gradually increase the value of x we can see that the deviation of P from the x axis and that of Q from y axis are same. So the y coordinate of P and the x coordinate of Q are same (except in sign - may be). So we get the result $\cos(n\pi \pm x) = \pm \sin x$. Congruency of the triangles gives the other result $\sin(n\pi \pm x) = \pm \cos x$.

3. Any real number y can be written in the form $y = \frac{n\pi}{2} \pm x$ for some non negative integer n and $0 \leq x \leq \frac{\pi}{2}$. So the behaviour of trigonometric functions in the set of real numbers can be studied by observing them in $[0, \frac{\pi}{2}]$

Additional Activities

Activity 5.A $\cos(x + y)$

Derives the trigonometric identity using the applet	Triangles are congruent. So AC = BP. Derive the result using distance formula.	Gets the result $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

Activity 5.B Simple Harmonic Motion

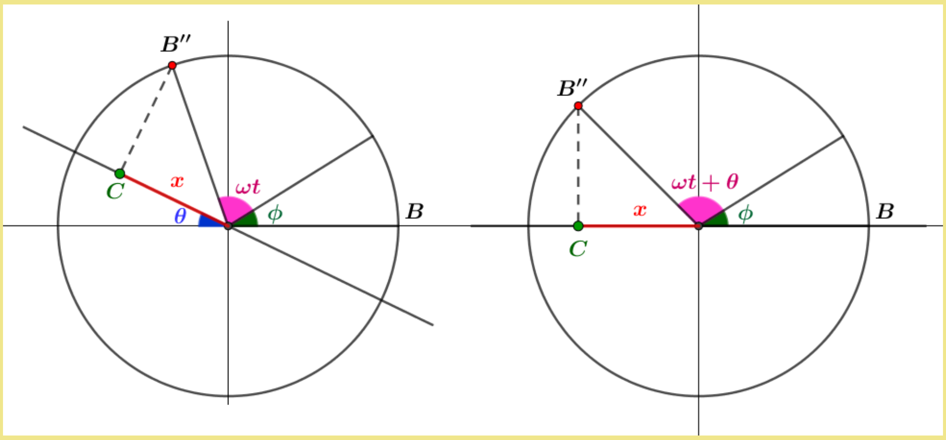
In this activity we create a point that moves along a straight line in SHM. For this we use the concept that an SHM as a projection of a uniform circular motion on a straight line. We use the concept trigonometric functions to derive the equation of motion of a body in SHM.

☛ Creates the applet ☛ Observes the movement of C	Discuss the derivation of equation of motion of a particle in SHM [1]	Derives the equation of motion of SHM

1. ☛ We take the projection of B'' on x axis. The rotation of B'' from B is $\omega t + \phi$. So the x coordinate of B'' and hence that of C is $A\cos(\omega t + \phi)$. So the distance of C from the origin at time t, that is $x(t)$ is $A\cos(\omega t + \phi)$. That is the equation of motion of C is $x(t) = A\cos(\omega t + \phi)$.

   ☛ If we take the projection to y axis the equation becomes $x(t) = A\sin(\omega t + \phi)$. But, by replacing ϕ by $\phi - \frac{\pi}{2}$ we can rewrite the equation as $x(t) = A\cos(\omega t + \phi)$.

   ☛ In general, suppose we take the projection of B'' on a line which makes an angle θ with x axis (as shown in the first figure). Let the displacement of C at time t is x.



   Rotate B'' and the line by an angle θ so that the line coincides with the x axis (second figure). Now the rotation of B'' from B is $\omega t + \theta + \phi$. So $x = A\cos(\omega t + \theta + \phi)$ is of the form $x = A\cos(\omega t + \phi)$. Thus we can conclude that the general equation of SHM is $x(t) = A\cos(\omega t + \phi)$.

Lab 6

Solutions of trigonometric equations

This lab consists of four activities through which we discuss the principal and general solutions of Trigonometric functions.

Required Concepts

- Principal and general solutions of trigonometric equations.

Aim

- In these activities students find principal and general solutions of trigonometric equations with the help of graphs. They get more clarity of the concepts by these activities

Activity 6.1 Solution of $\sin x = a$

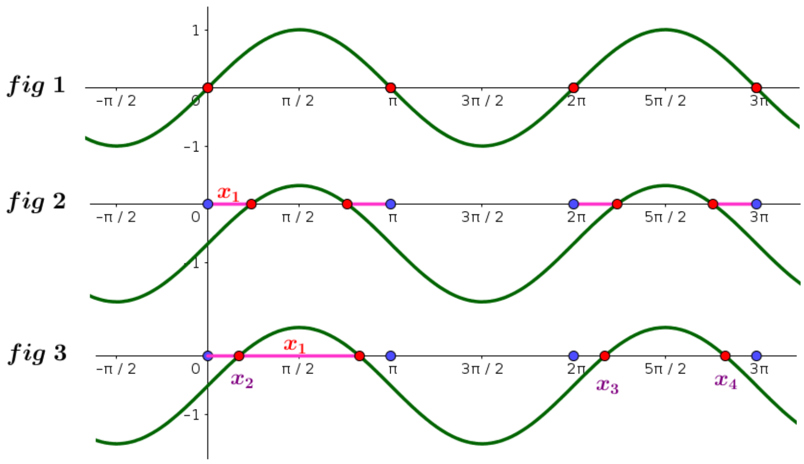
• Observes the points at which the graph of $\sin x - a$ cuts or touches the x axis • Observes the deviation of above points, as **a** changes, from the multiples of π	Discuss the general solution in terms of a principal solution using the dynamism of the applet [1]	• Gets more clarity on the concepts of principal and general solutions • If x_1 is a principal solution of the equation $\sin x = a$ then the general solution is $x = n\pi + (-1)^n x_1$ • Completes the table

1. When $a = 0$ the graph meets the x axis at multiples of π (fig 1). As we increase the value of a points of intersection deviates,by the same distance, from the multiples of π.

   If we take principal solution x_1 shown in fig 2 we can see that the points of intersections deviates towards left from odd multiples of π and towards left from even multiples of π by x_1.

   If we take principal solution x_1 shown in fig 3, then $x_2 = \pi - x_1, x_1 = 0\pi + x_1 x_3 = 3\pi - x_1, x_4 = 2\pi + x_1$ *etc.*

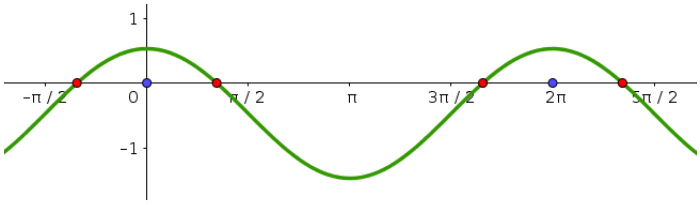
   In both cases we can write the general solution as $x = n\pi + (-1)^n x_1$



Activity 6.2 Solution of $\cos x = a$

Observes the points at which the graph of $\cos x - a$ cuts or touches the x axis	Discuss the general solution in terms of a principal solution using the dynamism of the applet [1]	☛ Gets more clarity on the concepts of principal and general solutions ☛ If x_1 is a principal solution of the equation $\cos x = a$ then the general solution is $x = 2n\pi \pm x_1$ ☛ Completes the table

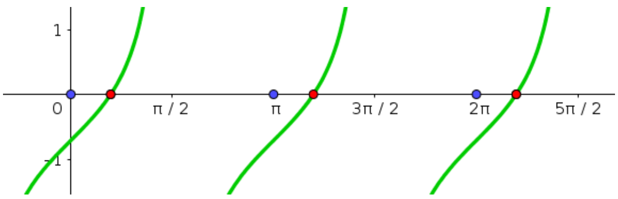
1. Set the value of a as 1. Then the graph touches the x axis on even multiples of π. As we decrease the value of a we can see that the points of intersection of the graph with the x axis deviates from even multiples of π towards both sides. This implies that the general solution is $x = 2n\pi \pm x_1$, where x_1 is any principal solution.



Activity 6.3 Solution of $\tan x = a$

Observes the points at which the graph of $\tan x - a$ cuts or touches the x axis	Discuss the general solution in terms of a principal solution using the dynamism of the applet [1]	☛ Gets more clarity on the concepts of principal and general solutions ☛ If x_1 is a principal solution of the equation $\tan x = a$ then the general solution is $x = n\pi + x_1$ ☛ Completes the table

1. Set the value of a as 0. Then the graph cuts the x axis on multiples of π. As we increase the value of a we can see that the points of intersection of the graph with the x axis deviates towards right (if we decrease a the deviation will be towards left) from multiples of π by x_1. This implies that the general solution is $x = n\pi + x_1$, where x_1 is any principal solution.



Activity 6.4 Solution of trigonometric equations in general

Observes the points at which the graph of $f(x) - g(x)$ cuts or touches the x axis	☛ Discuss the the method of finding solutions ☛ Also discuss the method of finding solution of the equation $f(x) = g(x)$ from the point of intersection of the graphs of $f(x)$ and $g(x)$ [1]	Completes the table

1. In some situations it is more convenient to visualise the solutions as the point of intersections of the graphs of $f(x)$ and $g(x)$. For example, to find the solutions of the equation $\sin x = \cos x$, since the students are familiar with the graphs of $\sin x$ and $\cos x$, they can visualise the graphs and their points of intersection in mind and easily find the solutions. On the other hand it may not be easy to deal with the function $\sin x - \cos x$ because the students are not familiar with its graph.

Lab 7

Shifting and scaling of graphs of trigonometric functions

In this lab we discuss the trigonometric identities like $\sin(\frac{\pi}{2}+x) = \cos x$, $\cos(\pi+x) = -\cos x$ etc. graphically. We also discuss the periodicity of trigonometric functions and shifting and scaling of graph of trigonometric functions. In the additional activities we discuss the applications of trigonometric functions in some physical situations like study of waves, music etc.

Required Concepts

☛ Graph of trigonometric functions
☛ Periodicity of a function
☛ Shifting of graphs

Aim

☛ To deduce the trigonometric identities like $\sin(\frac{\pi}{2}+x) = \cos x$, $\cos(\pi+x) = -\cos x$ etc. graphically
☛ To study the effect of the constants a, b and c of the trigonometric function $a\sin(bx+c)$ on the graph of the function. Hence deduce the period of trigonometric functions.

Activity 7.1 Shifting

Observes the change in the graph of $\sin(x+a)$ according to a	☛ Recall the shifting of graphs discussed in Lab 2, Activity 2.2 ☛ Discuss the periodicity of a function ☛ Ask the students to guess the period of $\sin x$ and then verify it with GeoGebra. [1]	☛ Gets a clear idea about the periodicity of trigonometric functions. ☛ Finds the period of $\sin x$ (2π)
Draws the graphs in Set-1, Set-2, and Set-3 and compare	☛ Ask the students to guess the graphs of the functions given in Set-1 and Set-3 using the concept of shifting and compare them with the graphs of the functions given in Set-2.(Do all these in mind). Then verify their result using GeoGebra. ☛ Discuss the results in general. [2]	☛ Derives the reduced forms ☛ Completes the tables ☛ Understands the method of finding the reduced forms in general

1. By observing the value of the slider students may conclude that the period of $\sin x$ is 6.28. Make it clear that it is an approximate value of 2π. Give input 2pi in the input box for the slider to get the exact graph.

2. ☛ $\sin(n\pi \pm x) = \pm \sin x$, $\cos(n\pi \pm x) = \pm \cos x$ and $\tan(n\pi \pm x) = \pm \tan x$. The sign on the right hand side is determined by the quadrant in which $n\pi \pm x$ lies.
   For example $\sin(5\pi + x) = -\sin x$, because $5\pi + x$ lies in the third quadrant where sine is negative
   ☛ If n is odd, $\sin(n\frac{\pi}{2} \pm x) = \pm \cos x$, $\cos(n\frac{\pi}{2} \pm x) = \pm \sin x$ and $\tan(n\frac{\pi}{2} \pm x) = \pm \cot x$. Here also the sign on the right hand side is determined by the quadrant in which $n\frac{\pi}{2} \pm x$ lies.
   For example $\sin(3\frac{\pi}{2} + x) = -\cos x$, because $3\frac{\pi}{2} + x$ lies in the forth quadrant where sine is negative.
   ☛ Remark : Above discussion is based on the assumption that $0 < x < \frac{\pi}{2}$. But the result holds for any real value of x.
   ☛ Above results are discussed in Lab 5 and their geometrical proof is discussed in Activity 5.3

Activity 7.2 Scaling

Draws the graphs of $a \sin x$ and $a \cos x$ and observes their change according to a	☛ A vertical elongation/compression occurs to the graph of the functions as a increases/decreases. ☛ Domain of $a \sin x$ and $a \cos x$ are independent of a (that is domain is the set of real numbers) and their range is $[-a, a]$	☛ Understands the change in the graphs of the functions $a \sin x$ and $a \cos x$ ☛ Finds the domain and range of above functions

Activity 7.3 Periods of Trigonometric Functions

Draws the graphs of $\sin nx$ and $\cos nx$ and observes their change according to n	☛ A horizontal compression occurs to the graph of the functions as n increases. And a horizontal elongation occurs as n decreases. ☛ Discuss the period of the functions. [1]	☛ Understands the change in the graphs of the functions $\sin nx$ and $\cos nx$ according to n ☛ Finds the period of above functions
Draws the graphs of $\sin \frac{x}{n}$ and $\cos \frac{x}{n}$ and observes their change according to n	☛ A horizontal elongation occurs to the graph of the functions as n increases. And a horizontal compression occurs as n decreases. ☛ Discuss the period of the functions.[2]	☛ Understands the change in the graphs of the functions $\sin \frac{x}{n}$ and $\cos \frac{x}{n}$ according to n ☛ Finds the period of above functions

1. ☛ The period of the functions decreases as n increases.
   ☛ For $n = 1$ the period of the function ($\sin x$ or $\cos x$) is 2π. As n changes to 2, the graph is compressed along x axis. So it oscillates two times between 0 and 2π, so its period is $\frac{2\pi}{2}$ that is π. In other words it completes one full oscillation between 0 and π. Similarly for $n = 3$ the period is $\frac{2\pi}{3}$ and in general the period of $\sin nx$ and $\cos nx$ is $\frac{2\pi}{n}$.

2. ☛ The period of the function increases as n increases.
   ☛ Period of the functions $\sin \frac{x}{n}$ and $\cos \frac{x}{n}$ is $2n\pi$

Activity 7.4 Shifting and Scaling

Draws the graphs of $a\sin(bx + c)$ and $a\cos(bx+c)$ and observes the effect of a, b and c on them.	☛ Recall the shifting, horizontal and vertical elongation/compression discussed in above activities ☛ Change in $a \Rightarrow$ vertical elongation/-compression by a times ☛ Change in $b \Rightarrow$ horizontal elongation/compression by b times ☛ Change in $c \Rightarrow$ horizontal shifting by c units	Understands the effect of a, b and c on the graphs of $a\sin(bx + c)$ and $a\cos(bx + c)$

Additional Activities

Activity 7.A Waves

In this activity we discuss `Waves` with the help of trigonometric functions.

From a physics perspective, wave motion is a transfer of energy and momentum without the shift of particles actually involved. The particles involved would oscillate with an amplitude a which is the maximum displacement of the particle from its equilibrium position. If the particles oscillates in the direction perpendicular to the direction of propagation of the wave, we call the wave as a transverse wave. If the particle oscillates in the same direction as that of wave propagation, we call the wave as a longitudinal wave. The ripples formed on the surface of water is an example of transverse wave and sound wave is an example of longitudinal wave.
From a Mathematical perspective, the displacement of a particle during the event of a wave can be represented as a sinusoidal function. Consider a wave propagating along the positive direction of the x axis. Consider a particle on x initially at a distance of x units from the origin. As the wave propagates along x axis, this wave oscillates about its initial position (either perpendicular or parallel to x axis). Its displacement from its initial position (y) depends on its initial position (x) and time t. It is given by

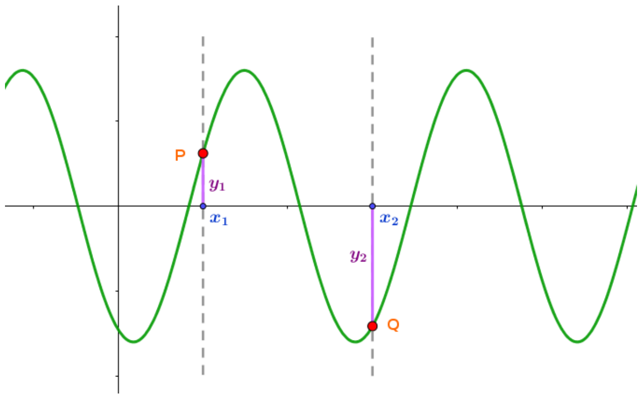
$y = f(x, t) = a\sin(kx - \omega t)$

where a is the amplitude, ω is the angular frequency and k is called the angular wave number or scaling factor.

Example of a transverse wave is shown in the following figure. Initial position of the points P and Q are x_1 and x_2 respectively and their displacement from their initial positions at time t are y_1 and y_2. Then

$y_1 = f(x_1, t) = a\sin(kx_1 - \omega t)$ and
$y_2 = f(x_2, t) = a\sin(kx_2 - \omega t)$

As the wave propagates along the positive direction of x axis all the particles on x axis oscillates like this.



We can show the oscillation of a particle on x axis using the applet created by the students as follows.
Take any point on x axis. Draw a line passing through this point and perpendicular to x axis. Plot the point of intersection of this line with the curve f. Give animation to the slider t. As t increases we can see that the wave moves along the positive direction of the x axis and the point of intersection oscillates perpendicular to the x axis.

The wave represented by the equation $g(x, t) = a\sin(kx + \omega t)$ propagates along the negative direction of the x axis. In physics this wave is called the reflection of the first wave.

$f + g$ represents the resultant of above two waves. Even though the particles vibrates, this wave doesn't move in any direction. Hence it is called a standing wave. We can observe standing wave on a vibrating stretched string.

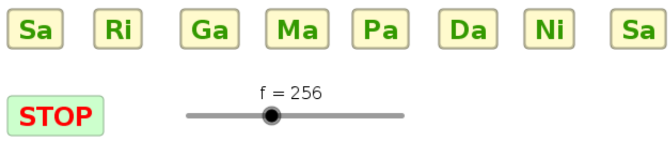
The applets MLHB7A1 and MLHB7A2 may be used to make above ideas more clear.

Activity 7.B Music and Maths

This activity will be interesting for the students those who have a taste in music.

We can play saptaswaras using GeoGebra with the help of PlaySound command as follows

For adjusting the base frequency create an integer slider f with Min=100 and Max=500.
Create 8 Buttons named as Sa, Ri, Ga, Ma, Pa, Da, Ni and Sa. Create one more slider to stop the sound.



Create a function named as 'Sa' by giving input Sa=sin(f 2pi). Similarly create the following functions.
Ri=sin(16f/15 2pi)
Ga=sin(5f/4 2pi)
Ma=sin(4f/3 2pi)
Pa=sin(3f/2 2pi)
Da=sin(8f/5 2pi)
Ni=sin(15f/8 2pi)
Sa2=sin(2f 2pi)

For the Button 'Sa' , Write the command as PlaySound[Sa,0,5] in the Scripting tab. For the Button 'Ri' Write the command as PlaySound[Ri,0,5] and so on. Also write the command PlaySound[False] for the stop button.

Clicking on the first 'Sa' we can hear a sound of frequency f, which is taken as the base note 'Sa' (Shadja). Frequencies of all other swaras are defined in terms of this base note. Frequency of 'Ri' is $\frac{16}{15}$ times the frequency of 'Sa', that is $\frac{16}{15}f$, frequency of 'Ga' is $\frac{5}{4}f$ and so on. The freequency of the second 'Sa' is twice the frequency of the first one.

Note: Here we discuss the notes of the raga 'MayamalavaGaula' with which music students usually starts their lessons. For other ragas the frequencies of some swaras, except that of 'Sa' and 'Pa', may differ.

By clicking on the swara buttons we can play saptaswaras - the notes of 'MayamalavaGaula'

The applet MLHB7B may be used.

Activity 7.C Harmonic Sound

Beats and harmonic sounds are discussed in Higher Secondary physics class. If a note of frequency f is taken as the base note, then a note of frequency $2f$ is called the first harmonic and that of $3f$ is called the second harmonic and so on. If we hear a note with its harmonics, we hear a sound with the base frequency. If we slightly change the frequency of any one of the notes, we can hear a beat.

The applet MLHB7C may be used.

Activity 7.D Blood Pressure

This activity relates trigonometric functions with another practical problem.

The period of $f(x) = sin(nx)$ is $\frac{2\pi}{n}$. So the period of $p(t) = 115 + 25\sin(160\pi t)$ is $\frac{2\pi}{160\pi} = \frac{1}{80}$

Number of heart beats per minutes is the frequency of the function, which is the reciprocal of the period. Hence the number of heart beats is 80.

Since the maximum value of sine function is 1 and the minimum value is -1, the maximum value of the BP is 115+25 = 140 and the minimum value is 115-25 = 90. That is the blood pressure of the person is 140/90, which is higher than the normal value 120/80.

Lab 8

Straight lines

This lab consists of four activities. We discuss the change in the straight line according to the changes in the coefficients and constant term of its general equation. We also discuss family of straight lines, straight line passing through the intersection of two lines, normal form of a line and shifting of origin.

Required Concepts

- General equation of a straight line
- Family of straight lines
- Normal form of a straight line

Aim
In Activity 8.1 students construct a straight line in general form and establishes the effect of coefficients and constant term in the nature of the line. In Activity 8.2 we discuss the family of straight lines passing through the intersection of two lines. In Activity 8.4, geometry of the normal form of a straight line and in Activity 8.5 Shifting of origin are discussed.

Activity 8.1 General Form of Straight Lines

Observes the peculiarity of the line $ax + by + c = 0$ for particular values of a, b, and c such as $a = 0$, $b = 0$, $a = b$ etc.	Discuss the nature of the line for particular values of the coefficients and the constant in the equation of the line.[1]	Establishes the nature of the line for particular values of the coefficients and the constant in the equation of the line
Observes the movement of the line while changing the value of only one of a, b, and c	Discuss the peculiarity of family of lines obtained in each case [2]	Understands the concept of family of straight lines

1. ☛ $a = 0$: Line is parallel to x axis
   ☛ $b = 0$: Line is parallel to y axis
   ☛ $c = 0$: Line passes through the origin
   ☛ $a = b$: Line is parallel to the line $y = -x$, x, y intercepts are same.
   ☛ $a = -b$:Line is parallel to the line $y = x$, x, y intercepts are equal but opposite in sign.
   ☛ ☛ When a alone changes the line rotates about a point on y axis.
      ☛ When b alone changes the line rotates about a point on x axis.
      ☛ When c alone changes the line moves parallel to the initial line.

Activity 8.2 Intersection of Two Lines

Observes the movement of the line as k changes.	☛ Discuss the values of k for which the new line coincides with the given lines [1] ☛ Discuss the family of lines passing through the point of intersection of two lines [2] ☛ Discuss the equation of family of lines passing through a given point. [3]	Makes observations and understands the concept of family of lines passing through the point of intersection of two lines and a general equation to represent a member of that family.
Observes the movement of the third line according to k while the first two lines are parallel.	☛ Discuss the reason that the third line is parallel to the first two lines.[4] ☛ Discuss the values of k for which the new line coincides with the given lines [5]	Observes that the third line is parallel to the given lines.

1. ☛ When $k = 0$ the third line coincides with the first.
   ☛ As the value of k increases (to $\pm\infty$)third line approaches the second line. But it never coincides with it.

2. ☛ The equation $k_1(2x-3y+4)+k_2(2x+5y-6) = 0$ (a linear combination of the given equations) represents the family of all lines passing through the intersection of the given lines. If $k_1 \neq 0$ we can write above equation as $2x-3y+4+k(2x+5y-6) = 0$ (where $k = \frac{k_2}{k_1}$) which represent the family of all lines, except the second line, passing through the intersection of the given lines.

3. ☛ A linear combination of equations of any two lines passing through the given point (say (a, b))represents the family of lines passing through that point. For convenience consider the lines $x = a$ and $y = b$. Then the equation $k_1(x - a) + k_2(x - b) = 0$ represents the family of lines passing through that point (a, b).
   ☛ Ask the students to do this with GeoGebra.

4. ☛ Since the lines are parallel we can take their equations as $ax + by + c_1 = 0$ and $ax+by+c_2 = 0$. Their linear combination is $k_1(ax+by+c_1)+k_2(ax+by+c_2) = 0$. That is $(k_1 + k_2)(ax + by) + k_1c_1 + k_2c_2 = 0$. Which can be written in the form $ax + by + c = 0$

5. ☛ When $k = 0$ the third line coincides with the first.
   ☛ As the value of k increases (to $\pm\infty$)third line approaches the second line. But it never coincides with it.

Activity 8.3 Normal Form

Creates the applet and using it finds the normal form of the equations of the given lines	Discuss the method of finding normal form [1]	Makes observations and gets the normal form
For given values of ω and p finds the normal form of the lines and verifies the result using the applet.	☛ Discuss the method of finding the equations [2] ☛ Discuss the method of verification of the answer [3]	Completes the table.

1. Draw the given line using input box.Write the equations in normal form using the values of ω and p obtained from the figure.

2. Find the equations using $x \cos \omega + y \sin \omega = p$

3. Input the obtained equation in the input box of the applet to draw the line. Observe the values of ω and p and verify whether it matches with the given values.

Activity 8.4 Shifting of Origin

Shifts the origin, parallel to the x axis or y axis and observe the changes in the new equation	Discuss the change in the equation [1]	Makes observations and gets a clear idea about the transformed equation.
Guesses the new origin and verify it.	☛ Discuss the method of finding the new origin [2] ☛ Discuss the method of verification of the answer [3]	Completes the table.
Finds the transformed equation without using the applet and then verifies it using the applet.	Discuss the method mentioned in the text book for finding the transformed equation.	Understands the method of finding the transformed equation.

1. ☛ If the shift is parallel to x axis then only the terms of x changes in its equation and if the shift is parallel to y axis, terms of y only changes.
   ☛ If the origin is shifted horizontally by h units (towards right if h is positive and towards left if h is negative) then x becomes $x + h$.
   ☛ If the origin is shifted vertically by k units (upwards if k is positive and downwards if k is negative) then y becomes $y + k$.
   ☛ We can combine above two observations as follows.
   - ☛ If the origin is shifted from $(0, 0)$ to (h, k) then the transformed equation is obtained by replacing x with $x + h$ and y with $y + k$
   - ☛ If the origin is shifted from (h_1, k_1) to (h_2, k_2) then the transformed equation is obtained by replacing x with $x + h$ and y with $y + k$ where $h = h_2 - h_1$ and $k = k_2 - k_1$

2. ☛ Since the transformed equation is independent of x and y terms, new origin should be the centre of the circle, that is $(3, 3)$.

3. ☛ Shift the origin to the point obtained by the guess of the students and verify whether the transformed equation matches with the given equation.

Lab 9

Conic Sections

In the first activity, we identifies the curves Circle, Ellipse, Parabola and Hyperbola as the section of a cone. We study the curves as the locus of a point through the remaining four activities. Two additional activities are also discussed in this lab.

Required concepts

- Cone and its section by a plane
- Locus of a point

Aim

- Activity 9.1 deals with the conic section as the section of a cone. Students get a clear idea about the formation of the curves, how the curves are related with the semi vertical angle of the cone, the angle made by the cone with the axes of the cone and the position of the plane.
- In Activity 9.2 we discuss a simple example to get the concept of locus.
- Activities 9.3, 9.4 and 9.5 deals the curves as the locus of a point
- Focus directrix definition of ellipse and hyperbola is discussed in the additional Activity 9.A
- In Activity 9.B we discuss another example of locus of a point which gives a set of circles called Apollonius Circles.

Activity 9.1 Cutting of a Cone by a Plane

• Changes the value of β and observes the curve. • Repeats the activity for different values of α	• Help the students to make a general conclusion about the formation of the curves according to the values of the angles α and β. [1] • Discuss the change in shape of the curves according to the position of the plane. [2] • Discuss the change in the shape of the curve according to α [3]	• Completes the table • Makes a general conclusion about the formation of the curves according to the values of the angles α and β. • Understands the change in shape of the curves according to the position of the plane

1. ☛ The relation between the curves and angles is as follows
   - ☛ Circle $\rightarrow \beta = 90°$
   - ☛ Parabola $\rightarrow \beta = \alpha$
   - ☛ Ellipse $\rightarrow \alpha < \beta < 90°$
   - ☛ Hyperbola $\rightarrow 0 < \beta < \alpha$
   - ☛ If $\beta = 0$ we get a pair of straight lines

   ☛ It is enough to consider the acute angle made by the plane with the axis of the cone. So we take the values of β as $0 \leq \beta \leq 90°$.

2. ☛ Fix the angles α and β. Using slider move the plane from top to bottom. Observe the following changes of the curves.
   - ☛ Circle $\rightarrow$ Radius decreases, at the apex it becomes a point, then radius increases
   - ☛ Parabola $\rightarrow$ Width of the parabola decreases, at the apex it becomes a line (because it coincides with a generating line), then the curve gets inverted and its width increases as the plane moves downwards.
   - ☛ Ellipse $\rightarrow$ Ellipse becomes smaller and smaller, at the apex it becomes a point and then it increases its size.
   - ☛ Hyperbola $\rightarrow$ Distance between the branches decreases, at the apex it becomes a pair or intersecting lines, then the distance between the branches increases.

3. ☛ If $\beta = 90°$ then the radius of the circle increases as α increases.

   ☛ If $\beta < 90°$ then the curve changes from ellipse to parabola and then to hyperbola as α increases.

Activity 9.2 Locus of a point moving equidistant from given points

Creates the applet and observes the path of the moving point.	☛ Discuss the locus of a point [1] ☛ Discuss the reason that the path traced by the point is the perpendicular bisector of AB.[2]	☛ Understands the concept of Locus of a point ☛ Identifies that the path traced by the moving point is the perpendicular bisector of AB.

1. ☛ We can define the locus of a point (on a plane) in any one of the following ways.
   - ☛ The set of points on the plane that satisfy some specific conditions.
   - ☛ The path traced by a point which moves on the plane under some given conditions.

   ☛ In the second definition we feel a dynamism. So we follow it since we use dynamic applets to describe the curves.

2. ☛ In the applet we use circles of same radius centered at the points A and B. So their point of intersection is equidistant from these points. Hence it lie on the perpendicular bisector of AB.

Activity 9.3 Locus of a point the sum of whose distances from two given points is a constant

Creates the applet and observes the path	☛ Discuss the condition satisfied by the moving points. Hence derive the definition of ellipse. [1] ☛ Discuss the change in the shape of the curve according to the distance between A and B[2]	Understands the definition of Ellipse

1. The radius of the circle centered at A is a and that centered at B is $10 - a$. So sum of the distances of the moving point from A and B is 10.

2. ☛ As B approaches nearer to A, the curve becomes more circular
   ☛ When the distance between A and B is 10, the circles touches together and the point of contact is on the line segment AB. So the path traced is the line segment AB.
   ☛ When the distance between the points is greater than 10, the circles do not meet together, hence no curve is formed. (We can relate this with the algebraic equation of ellipse, in which $a > c$)

Activity 9.4 Locus of a point the difference of whose distances from two given points is a constant

Creates the applet and observes the path	☛ Discuss the condition satisfied by the moving points. Hence derive the definition of hyperbola.[1] ☛ Discuss the change in the shape of the curve according to the distance between A and B[2]	Understands the definition of Hyperbola

1. The radius of the circle centered at one point is a and that centered at the other point is $14 + a$. So difference of the distances of the moving point from A and B is 4.

2. ☛ As the distance between the points increases the width of the curve increases.
   ☛ When the distance between A and B is 4, the circles touches together and the point of contact is on the line AB outside the line segment AB. So the path traced is the line AB - segment AB.
   ☛ When the distance between the points is less than 4, the circles do not meet together, hence no curve is formed.

Activity 9.5 Locus of a point equidistant from a point and a fixed line

Creates the applet and observes the path	☛ Discuss the condition satisfied by the moving points. Hence derive the definition of Parabola.[1] ☛ Discuss the change in the shape of the curve according to the distance between the point and the line[2]	Understands the definition of Parabola

1. The moving line is parallel to the fixed line and is at a distance of a units from it. Radius of the circle centered at C is also a. Hence point of intersection of this circle with the moving line is equidistant from the fixed line and the point C.

2. Width of the curve increases as the point moves away from the fixed line.

Activity 9.A Focus - Directrix Definition

As in the case of Parabola we can define Ellipse and Hyperbola in terms of distance of a point from a fixed line and a fixed point. The fixed line is called directrix and the fixed point is called focus. The ratio of the distance of the moving point from the focus to its distance from the directrix is called the eccentricity (e) of the curve.

If $e = 1$ then the moving point is equidistant from the line and the point. Hence we get a Parabola.

If $e < 1$ then the locus of the point is an Ellipse and if $e > 1$ the locus is a Hyperbola.

In the applet, distance of the moving lines from the fixed line (directrix - y axis) is a and the radius of the circle centered at the fixed point (focus) is ba. So considering the point of intersection, ratio of its distance from the focus to the directrix is b. That is, in this discussion the eccentricity is b.

We get different curves according to the value of b as discussed above.

Activity 9.B Apollonius Circles

The radii of the circles are r and ar. So considering the point of intersection of the circles, the ratio of its distance from B to that from A is r.

If $r = 1$, then the point of intersection is equidistant from A and B and the locus will be the perpendicular bisector of AB.

If $r \neq 1$ the locus will be a circle, called Apollonius circle.

The circle meets the line AB at two points. One of the points divides AB in the ratio $1 : r$ internally and the other point divides it in the same ratio externally.

Lab 10

Circle and Parabola

This lab consists of 3 activities and two additional activities. Here we discuss different methods of drawing ellipse and hyperbola using GeoGebra tools, commands and equations. In additional activities we create some interesting patterns using circles. We also discuss a method of drawing parabola as the locus of a point.

Required concepts

- Definitions of Circle and Parabola
- Equations of Circle and Parabola

Aim

- By this activities students get a clear idea about equation of a circle,centre,focus and latus rectum of a Parabola
- Students realise the role of above parameters in determining the shape of the curve
- For drawing a circle or a parabola using a specific tool they may have to do some algebraic calculations or to think geometrically. This give a thorough understanding about the curves and their equations.

Activity 10.1 Circle

Finds the centre and radius of the circles algebraically and verifies the results geometrically.	☛ Recall the methods of finding centre and radius of a circle from its equation ☛ Discuss different methods of geometrical verification (given in the Lab Manual)	Gets the centre and radius of the circles
Finds the equation of the circles algebraically and verifies geometrically	☛ Recall the method of finding equations. ☛ Discuss the method of verification.	Gets the equations of the circles
Finds the geometrical methods of construction	☛ Discuss the common logic behind the three problems. [1] ☛ Discuss the geometric method of construction of the circles. [2] ☛ Discuss the algebraic method for finding the equation of a circle passing through three points. [3]	☛ Constructs the circles ☛ Derives an algebraic method for finding the equation of a circle passing through three points.

1. If A and B are two points on a circle, then AB is a chord and the centre of the circle lies on its perpendicular bisector

2. ☛ Problem 1 :
    - ☛ Plot the points $A(2, 3)$ and $B(-1, 1)$
    - ☛ Draw the line segment AB
    - ☛ Draw the line $x - 3y - 11 = 0$
    - ☛ Plot the point of intersection of above line with the perpendicular bisector.
    - ☛ Draw the circle centered at above point ad passing through A or B

   ☛ Problem 2 :
    - ☛ Plot the points $A(1, 2)$, $B(5, 4)$ and $C(3, 6)$
    - ☛ Draw the chords AB and BC
    - ☛ Draw the perpendicular bisectors of the chords and plot their point of intersection.
    - ☛ Draw the circle centered at above point and passing through any of the given points.

   ☛ Problem 3 :
    - ☛ Find the equations of perpendicular bisectors of any two chords.
    - ☛ Find the point of intersection of the perpendicular bisectors.This gives the centre of the circle
    - ☛ Calculate the radius
    - ☛ Find the equation

Activity 10.2 Parabola 1

Draws the parabola and observes the change in its shape according to the distance between the line and point.	☛ Discuss different methods of drawing parabolas. ☛ The curve becomes more and more wide as the distance between the point and line increases.	☛ Draws the parabola ☛ Realise the fact that the distance between the focus and directrix determines the shape of the curve.
Finds the focus and directrix of the parabolas algebraically and verifies geometrically	☛ Recall the method of finding focus and directrix algebraically ☛ Discuss the method of verification (given in the Lab Manual)	Gets the focus and directrix of the parabolas.

Activity 10.3 Parabola 2

Draws the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ and observes the change in its shape according to a.	☛ Recall the effect of the distance between the focus and directrix on the shape of the parabola, discussed in the above activity 'Parabola 1'. ☛ The curve becomes more and more wide as a increases. It approaches to the y axis as $a \to \pm\infty$	☛ Understands the effect of the parameter a on the shape of the curve. ☛ Understands the fact that a is the one and only one parameter that determines the parabolas with vertex at the origin and axes along the coordinate axes.
Find the focus and length of latus rectum of the parabolas algebraically and verifies the answer geometrically.	☛ Recall the method of finding focus and length of latus rectum algebraically ☛ Discuss the method of verification (given in the Lab Manual)	Gets the focus and length of latus rectum of the parabolas.

Activity 10.A Family of Circles

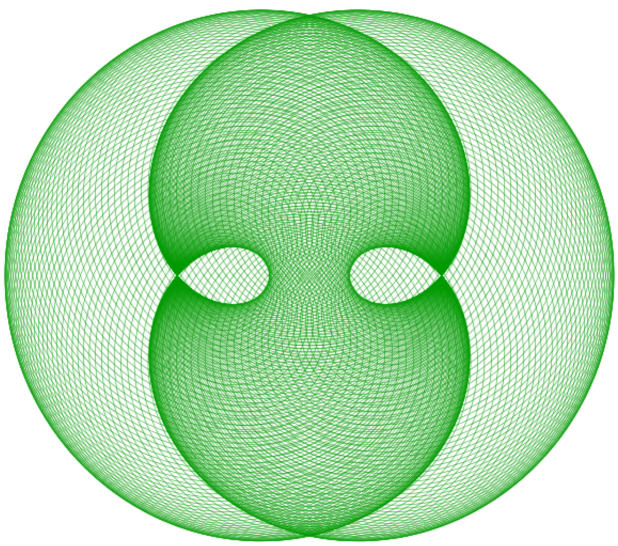
In this activity we draw some interesting patterns using circles with the help of 'Sequence' command.

Consider the command `Sequence[(x-r)^2 +y^2=r^2,r,0,3,.1]` . The equation of a general circle in this command is $(x-r)^2 + y^2 = r^2$. Its centre is (r,0), on the x axis and radius is r. So it passes through the origin. Hence the command gives a set of circles centered on the x axis and passing through the origin. (All the circles lie on the positive side of x axis because r varies from 0 to 3.
Similarly the command `Sequence[x^2 +(y+r)^2 = r^2,r,0,3,.1]` gives a set of circles centered

on the y axis and passing through the origin. (All the circles lie on the negative side of y axis).

Consider the following family of circles.

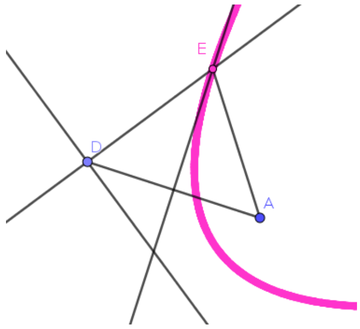
- Centres lie on the line y = x, and pass through the origin
  - Center can be taken as (a, a). Then the radius is $\sqrt{2}a$ (because the circle passes through the origin)
  - The command `Sequence[(x-a)^2+(y-a)^2=2a^2,a,-3,3,.1]` gives the circles
- Family of 100 circles of radius 3, whose centres lie at equal distance on the circle of radius 3 centred at the origin
  - Consider a circle of radius 3 centered at the origin.
  - The points $(3\cos(r\frac{2\pi}{100}, 3\sin(r\frac{2\pi}{100}),\ r = 0, 1, 2, ..., 99$ are 100 equidistant points lie on the above circle.
  - The command `Sequence[Circle[(3cos(r*2pi/100),3sin(r*2pi/100)),3],r,0,99]` gives the required circles. (In above command we use `Circle(Point,radius)` command inside the sequence command)
  - The command `Sequence[(x-3cos(r*2pi/100))^2+(y-3sin(r*2pi/100))^2=9,r,0,99]` also gives the pattern. (Here we use the equation of the circle inside the sequence command)
- Family of 100 circles, whose centers lie at equidistant points on the circle of radius 3 centred at the origin and passing through the point (3, 0)
  - The command `Sequence[Circle[(3cos(r*2pi/100),3sin(r*2pi/100)),(3,0)],r,0,99]` gives the required circles. (In above command we use `Circle(Point,Point)` command inside the sequence command)
- Do the above activity with a slider n to change the number of circles and another slider a so that all the circles pass through (a, 0) or (0, a) instead of (3, 0).
  - Use the command `Sequence[Circle[(3cos(r*2pi/n),3sin(r*2pi/n)),(a,0)],r,0,n-1]` to get the family of circles passing through $(a, 0)$
  - `Sequence[Circle[(3cos(r*2pi/n),3sin(r*2pi/n)),(-a,0)],r,0,n-1]` gives the family of circles passing through $(-a, 0)$



Activity 10.B Parabola with Given Focus and Directrix

Discuss the reason that the curve traced is a parabola.

Since E is a point on the perpendicular bisector of AD, ADE is an isosceles triangle. So $ED = EA$. Since the line DE is perpendicular to the given line, DE is the distance of E from the given line. So $ED = EA \Rightarrow E$ is equidistant from the line and the point A. Hence the path of E is the parabola with A as the focus and the given line as the directrix.



Lab 11

Ellipse and Hyperbola

This lab consists of 4 activities and two additional activities. Here we discuss different methods of drawing ellipse and hyperbola using GeoGebra tools, commands and equations. In additional activities we discuss a little more general case of conic sections, that is Parabola Ellipse and Hyperbola whose axes are parallel to the coordinate axes. We also create an applet to visualise a problem given in the text book.

Required concepts

- Definitions of ellipse and hyperbola
- Equations of ellipse and hyperbola

Aim

- By this activities students get a clear idea about foci, major axis, minor axis, latus rectum of Ellipse. Similarly in the case of Hyperbola
- Students realise the role of above parameters in determining the shape of the curves
- For drawing an Ellipse or Hyperbola using a specific tool they have to use the relation between different parameters of the curve. This give a thorough understanding on it
- Students get a practice of finding different parameters related with an Ellipse or Hyperbola from their equations.

Activity 11.1 Ellipse 1

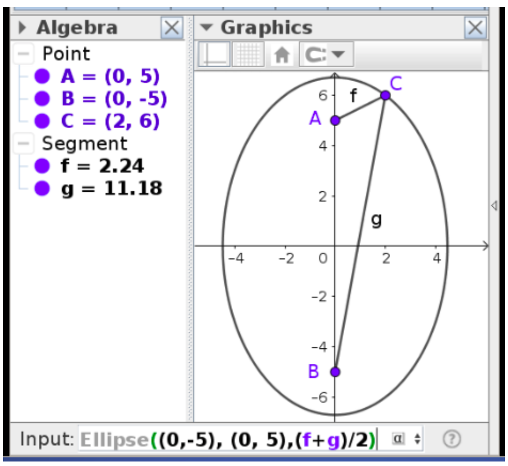
Using 'Ellipse' tool or the given input command draws the curves	• In some problems, foci or a point on the Ellipse are not given. Discuss different methods of finding them using given data [1] • Discuss the method of finding the length of the latus rectum geometrically [2]	• Draws the curves and saves the file • Finds the lengths of latus rectum geometrically and verifies the answer algebraically

1. 2. length of major axis 10 and major axis of the Ellipse is y axis. So $(0, 5)$ is a point on the Ellipse (an end point of the major axis)
   3. length of minor axis 5 and major axis of the Ellipse is x axis. So $(0, 2.5)$ is a point on the Ellipse
   4. $\mathbf{a} = 5$ and $\mathbf{b} = 4 \Rightarrow \mathbf{c} = 3$. So foci are $(0, \pm 3)$ and $(\pm 4, 0)$, $(0, \pm 5)$ are points on the Ellipse
2. First of all draw a latus rectum. For this, draw a line perpendicular to the major axis and passing through one of the foci. Plot the points of intersection of this line with the curve. Hide the line and draw the line segment join above points. Using 'Distance or Length' tool measure the length of the latus rectum.

Activity 11.2 Ellipse 2

☛ Draws the curves using foci and length of semi major axis ☛ Observes the change in the shape of the curve according to the change in the distance between the foci.	☛ In some problems, foci or length of semi major axis are not given. Discuss different methods of finding them using given data [1] ☛ Discuss the special cases of Ellipse [2]	☛ Draws the curves and saves the file ☛ Identifies the change in the shape of the curve according to the change in the distance between the foci.

1. 1. $\mathbf{a} = 5$, $\mathbf{b} = 3 \Rightarrow \mathbf{c} = 4$.
   2. Sum of distances of the point $(2, 6)$ from the foci is **2a**. We can find it either using distance formula or geometrically using Geogebra. For finding it geometrically, plot the points $A(0, 5)$, $B(0, -5)$ and $C(2, 6)$. Draw the segments AC and BC. In the Algebra view we can see the names of the segments (f and g) which also represent the lengths of the segments. So $\frac{f+g}{2} = 2a$. The input command **Ellipse[(0,-5),(0,5),(f+g)/2]** gives the required Ellipse.



2. If we decrease the distance between the foci shape of the ellipse becomes more and more circular and when the foci coincides each other it becomes a circle.(Make a comparison of this with the situation that we have discussed in **Activity 9.1**.That is as the value of β increases to 90° shape of ellipse becomes more and more circular and when $\beta = 90^\circ$ it becomes a circle.

   When the distance between foci becomes the length of major axis, ellipse becomes a line segment.

Activity 11.3 Hyperbola 1

Using 'Hyperbola' tool or the input command draws the Hyperbolas.	☛ In some problems, foci or a point on the Hyperbola are not given. Discuss different methods of finding them using given data [1] ☛ Discuss the method of finding the length of the latus rectum geometrically	☛ Draws the curves and saves the file ☛ Finds the lengths of latus rectum geometrically and verifies the answer algebraically

1. Same as in the case of Ellipse, discussed in **Activity 11.1**

Activity 11.4 Hyperbola 2

Students' Activity	Teachers' role	Students' Response/Findings
Draws the curves using foci and length of transverse axis	In some problems, foci or length of transverse axis are not given. Discuss different methods of finding them using given data [1]	Draws the curves and saves the file

1. 1. $\mathbf{a} = 5$, $\mathbf{b} = 3 \Rightarrow \mathbf{c} = \sqrt{34}$. Length of transverse axis is 10

   2. Difference of the distances of the point $(2, 6)$ from foci is **2a**. We can find it either using distance formula or geometrically using GeoGebra as in the case of ellipse discussed in the above activity.

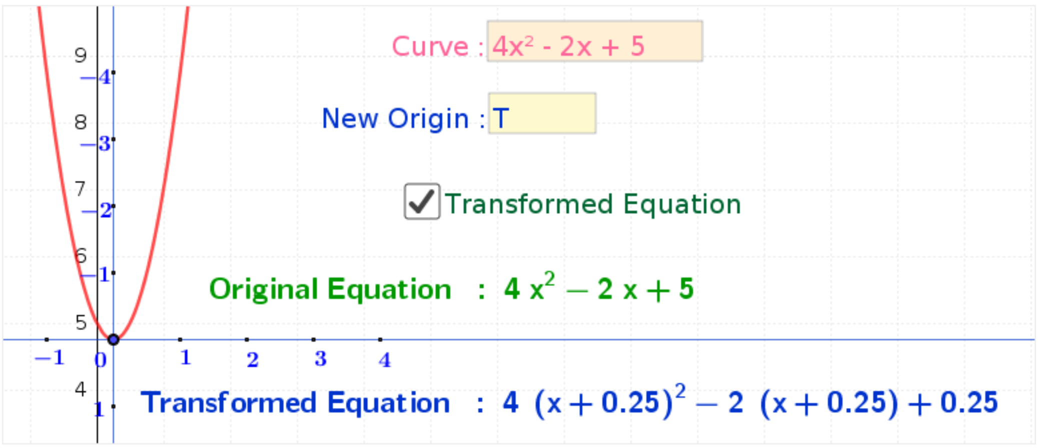
Activity 11.A Conic Sections in General

Solves the problems using the concept of Shifting of Origin.	Recalls the concept of Shifting of Origin. Discuss this activities as an application of Shifting of Origin.	☛ Understands an application of Shifting of Origin. ☛ Understands general equations of Parabola, Ellipse and Hyperbola with axes parallel to the coordinate axes

Note

If it is difficult to shift the origin to the vertex by mere observation, the input command `Vertex(y=4x^2-2x+5)` gives the vertex T((0.25, 4.25)). Enter the name of the vertex (T) in the box 'New Origin'.

We get the transformed equation as $y = 4(x + 0.25)^2 - 2(x + 0.25) + 0.25$. That is $y = 4x^2$ or $x^2 = \frac{1}{4}y$ (we get this even without simplification. Because with respect to the new system of axes equation of the Parabola is in the form $x^2 = 4ay$). So w.r.to New system, Focus of the Parabola is $(0, \frac{1}{16})$ and directrix is $y = -\frac{1}{16}$. Now we can find the focus and directrix w.r.to the original system.($(\frac{1}{4}, 4\frac{5}{16})$ and $y = 4\frac{3}{16}$)



Open ML 8.4 applet and draw the ellipse. We gets its equation in the standard form. Shift the origin to the centre of the ellipse and observe the transformed equation. Observe that the transformed equation is independent of the terms of x and y. Discuss a method of finding equation of such curves algebraically.

Open ML 8.4 applet and draw the hyperbola. We gets its equation in the standard form. Shift the origin to the centre of the hyperbola and observe the transformed equation. Observe that the transformed equation is independent of the terms of x and y.

Activity 11.B Locus of a Point on a sliding Rod

This activity is the visualisation of a problem given in the Text Book (In the Text Book the length of the rod is 15. But here we take it as 6 for the convenience of drawing)

Creates the applet and observes the path of C	Discuss the path of C when it is near to A, near to B, at the midpoint of AB, at A and at B [1]	Identifies that the path of C is ellipse, circle or line segment according to its position.

1. Path of C according to its position is as follows

   Near to A → Ellipse with major axis along x axis
   Near to B → Ellipse with major axis along y axis
   Midpoint of AB → Circle
   At A or B → Line segments

Lab 12

Basics of 3D

This lab consists of four activities and three additional activities. We discuss the basics of three dimensional geometry and its simple applications such as construction of rectangular box, platonic solids etc.

Required concepts

- Coordinate axes and coordinate planes
- Coordinates of a point in space
- Octants
- Internal / External division

Aim

- Students get a clear idea about octants, properties of points on different octants, coordinate planes and coordinate axes and internal and external division.

Activity 12.1 Octants

Finds some points lie on coordinate axes, coordinate planes and in different octants.	• Help the students to watch the coordinate axes, coordinate planes and all octants with the help of `Rotate` tool. • Discuss the properties of points lie on coordinate axes, coordinate planes and in different octants. [1] • Discuss the naming systems of octants. [2]	• Gets a clear idea about the octants • Understands the properties of points lie on coordinate axes, coordinate planes and in different octants.

1. ☛ If a point lies on a coordinate axes, at most one of its coordinates is non zero. For example, a point on x axis is of the form $(a, 0, 0)$, a may or may not be equal to zero.
   ☛ If a point lie on a coordinate plane, atleast one of its coordinate is zero. For example, a point on xy plane is of the form (a, b, c), a and b may or may not be zero.
   ☛ If a point lie in any one of the octants, all of its coordinates are non zero.
   ☛ The relation between number of an octant and sign of coordinates of its points are as follows.
2. ☛ For the first four octants, z coordinate is positive and sign of x and y coordinates are same as the sign of first for quadrants.
   ☛ For the next four octants (5 to 8) z coordinate is negative and sign of x and y coordinates are same as the sign of first for quadrants
   ☛ For example, $(-2, 3, 4)$ is a point in the second octant and $(2, -3, -4)$ is in the eighth octant

Activity 12.2 Movement of a Point

Move the sliders according to the instructions given and observe the movements of the points	☛ Help the students to complete the task [1] ☛ Help the students to make general conclusions [2]	☛ Completes the table ☛ Makes general conclusions about movement of points in space

1. Let the students imagine the movement of the points and then verify with the applet.

Sl. No	Movement and value of sliders	Movement of the point
1	$\mathbf{y}_1 = 0$, $\mathbf{z}_1 = 0$ and move $\mathbf{x}_1$	moves along the x axis
2	$\mathbf{y}_1$ and $\mathbf{z}_1$ are any constants, move $\mathbf{x}_1$	moves parallel to x axis
3	$\mathbf{x}_1$ and $\mathbf{z}_1$ are any constants, move $\mathbf{y}_1$	moves parallel to y axis
4	$\mathbf{y}_1$ and $\mathbf{x}_1$ are any constants, move $\mathbf{z}_1$	moves parallel to z axis
5	$\mathbf{z}_1 = 0$, move $\mathbf{x}_1$ and $\mathbf{y}_1$	moves along xy plane [1]
6	$\mathbf{z}_1 = 2$, move $\mathbf{x}_1$ and $\mathbf{y}_1$	moves parallel to xy plane, at a distance of 2 units from it
7	$\mathbf{y}_1 = 0$, move $\mathbf{x}_1$ and $\mathbf{z}_1$	moves along xz plane
8	$\mathbf{y}_1$ any constant, move $\mathbf{x}_1$ and $\mathbf{z}_1$	moves parallel to xz plane
9	$\mathbf{z}_1$ any constant, move $\mathbf{x}_1$ and $\mathbf{y}_1$	moves parallel to xy plane

1) While animating x_1 and y_1, sometimes z_1 may also be animated. In that case, right click on z_1, stop animation, set its value as 0, and use Ctrl+F to erase the unwanted path already traced. Allow a little bit time to get a clear idea about the path. Use Rotate tool to see the path from a convenient angle.

2. ☛ If two coordinates are fixed and only one coordinate is changed the point moves parallel to one of the coordinates axes.
   ☛ If one coordinate is fixed and two coordinate are changed the point moves parallel to one of the coordinates planes. The distance of this plane from the coordinate plane is the absolute value of the fixed coordinate.

Activity 12.3 Box

Constructs the boxes.	☛ Let the students imagine the coordinates of the vertices of the top of the box and then verify it from the algebra window. ☛ Discuss the method of construction of the box whose vertices lie in different octants. [1]	☛ Constructs the boxes. ☛ Gets a clear idea about the octants and the properties of points lie in different octants.

1. It is easy to find four points in four different quadrants, that forms the vertices of a rectangle. Say $(3,2), (-1,2), (-1,-2)$ and $(3,-2)$. By shifting all these points along the negative direction of the z axis we get four points in four different octants (5,6,7 and 8) that forms the vertices of a rectangle. So the points are like this, $(3,2,-1), (-1,2,-1), (-1,-2,-1)$ and $(3,-2,-1)$. Draw the rectangle with these points as the vertices (plot the points and draw using Polygon tool or use the command Polygon[(3,2,-1),(-1,2,-1),(-1,-2,-1),(3,-2,-1)]. Chose a hight so that the vertices of the top lie in first four octants (say 4) and construct the prism with extrude tool.

Activity 12.4 Section of a Line by Coordinate Planes

Adjusts the coordinates get the required results.	☛ Discuss internal and external division of a line by a coordinate plane and the method of finding the answers. [1] ☛ Let the students note down each pair of coordinates in each case.	Gets a clear idea about internal and external division of a line by a coordinate plane

1. ☛ If the points A and B lie on different sides (half space) of a coordinate plane (say xy plane)then the z coordinates of the points are of different sign and the xy divides AB internally. If the z coordinates are of the same sign then the points lie on the same side of the xy plane and the division is external.
   ☛ If xy and yz planes divide line AB internally then z coordinates as well as x coordinates are of different sign. As an example consider the points (2,3,-4) and (-5,5,1).
   ☛ All the coordinates planes divides the line joining the points (3,-2,5) and (-4,3,-1) internally.

Additional Activities

Activity 12.A Construction of a Box by Cutting Squares from Corners and Folding up the Flaps

- ☛ Use a slider a with min $= 0$ and max $= 1.5$ to change the side of the squares cutting out from the corners of the card board.
- ☛ Define the vertices of the base of the box in terms of a as $(a,a), (5-a,a), (5-a,3-a)$ and $(a,3-a)$
- ☛ Draw the rectangle and construct the box
- ☛ Hide the top of the box
- ☛ Show the volume of the box
- ☛ Change the value of a, observe the volume. Find the maximum volume and the corresponding value of a.
- ☛ If we want to see the flaps, open the box using net tool. The rectangle corresponding to the top will also come, hide it.

Activity 12.B Platonic Solids

Construction of Platonic solids

Cube

Cube is the easiest platonic solids to construct. We can do it in different methods.
- Using Cube tool select any two points.
- Draw a square and Extrude tool

Regular Tetrahedron

Here also we can use different methods to construct
- Using Tetrahedron tool select any two points
- Draw a circle (of radius a - say) centered at the origin. Plot a point on the circle and rotate it by 120° and 240°. Complete the base triangle. Use Extrude to Pyramid tool, give the height as $a\sqrt{2}$ (we can calculate this using Pythagoras theorem).

Octahedron

We can consider an Octahedron as the join of two square pyramids. We use this idea to construct it.
- Draw a square of side 3 units
- Using Pythagoras theorem we can calculate the height of the pyramid as $1.5\sqrt{2}$
- Using Extrude to Pyramid tool click on the square and give the height as $-1.5\sqrt{2}$ (we use the negative sign to get the inverted pyramid)
- Using the same tool click again on the square and give the height as $1.5\sqrt{2}$ to get the other pyramid.

Dodecahedron

A Dodecahedron has 20 vertices. If we consider a Dodecahedron centered at the origin and suitably scaled and oriented we can take its coordinates as follows.

$(\pm 1, \pm 1, \pm 1)$, $(0, \pm\phi, \pm\frac{1}{\phi})$,$(\pm\phi, \pm\frac{1}{\phi}, 0)$, $(\pm\frac{1}{\phi}, 0, \pm\phi)$,

Where $\phi = \frac{1+\sqrt{5}}{2}$ (the golden ratio)

Plot above points and construct the faces using polygon tool.

If we want to change the size of the Dodecahedron, create a slider (say a) and multiply each coordinate of the vertices by a.

Icosahedron

Icosahedron has 12 vertices. Coordinates of an Icosahedron centered at the origin can be taken as $(0, \pm 1, \pm\phi)$, $(\pm 1, \pm\phi, 0)$ and $(\pm\phi, 0, \pm 1)$. As in the case of Dodecahedron, plot the points and construct the faces. If we want to change the size, create a slider (say a) and multiply each coordinate of the vertices by a.

Lab 13

Limits

In this lab we have five activities and one additional activity. We discuss the existence and different cases of non-resistance of limit geometrically.

Required Concepts

- ☛ Value of a function at a point
- ☛ Graph of a function

Aim

Through the activities we develop the concepts **Limit at a point, Left limit and Right limit**
We discuss some standard limits
We discuss different cases of non existence of Limits.

Activity 13.1 Geometrical Interpretation of Limits

Observes the coordinates A, B, A_2 and B_2 as $h \to 0$	☛ Recall that, if we call the x coordinates of A and B as x, then the y coordinates of A_2 and B_2 is $f(x)$ ☛ As $h \to 0$, $x \to 2$ and $f(x) \to 4$ ☛ Discuss the concept of limit at $x = 2$. That is as $x \to 2$, $f(x) \to 4$.[1] ☛ Introduce the notation of limit $\lim\limits_{x \to 2} f(x) = 4$	Understands the concept of limit of a function at a point.

1. To find the limit of a function f at a point a we can use the command Limit(f, a)

Activity 13.2 Limit of Rational Functions

Observes the co-ordinates of the points A_2 and B_2 as $h \to 0$ and finds the limit of the function at $x = 2$	☛ Why the points A_2 and B_2 vanish when $h = 0$? ☛ The fact that even though the function is not defined at $x = 2$ its limit exists at 2 is to be emphasised. [1] ☛ Discuss the existence of the limit of the function $y = \begin{cases} \frac{x^2 - 4}{x - 2} & \text{if } x \neq 2 \\ 6 & \text{if } x = 2 \end{cases}$ at $x = 2$. In this case the function is defined at $x = 2$ but the limit is not $f(2)$.	☛ Gets the value of the limit ☛ Realise the fact that a function may not be defined for the existence of its limit. ☛ Realise the fact that even though a function is defined at some point its limit may not be equal to the value of the function at that point.

1. In fact we need the concept of limit in such situations where we can't find something by direct methods. Some examples are given below.
   - ☛ Area of a circle is derived as the limiting case of area of regular polygon inscribed in it. (The applet MLHB13.2 may be used to describe this)
   - ☛ Instantaneous velocity is the limiting case of average velocities
   - ☛ Slope of tangent to a curve at a point is the limiting case of slope of secant lines

   Note :-[.3cm] To get more accurate values in the limiting process, edit the Min and Max values of the slider as 1.99 and 2.01 and increment as 0.001. Also set the rounding of decimal places to 5 digits (Options ⇒ Rounding ⇒ 5 decimal Places). For recording to spread sheet select 20 rows.

Activity 13.3 Limit of Piecewise Functions

Observes the co-ordinates of the points A_2 and B_2 as $h \to 0$	☛ Discuss the concept of left limit and right limit(in the given example, as $x \to 2$ from the left, $f(x) \to 4$ and as $x \to 2$ from the right, $f(x) \to 5$. Introduce the notations $\lim\limits_{x\to 2^-} f(x) = 4$ and $\lim\limits_{x\to 2^+} f(x) = 5$ ☛ If left limit $\neq$ right limit then limit doesn't exist at that point ☛ In the second problem, left limit and right limit coincides with 4. So $\lim\limits_{x\to 2} f(x) = 4$ ☛ If both left and right limits exist and are equal then the limit exist at that point	Understands the concept of left limit and right limit and existence of limit in terms of them

Discuss different cases of non existence of limits and nature of curve while analysing the problems

Function	Existence/Non existence of Limit	Nature of curve
$f(x) = \frac{1}{x}$, at $x = 0$	As $x \to 0$ from the right, $f(x)$ increases to ∞ and as $x \to 0$ from the left, $f(x)$ decreases to $-\infty$. Both left and right limits doesn't exist. So the limit doesn't exist at 0.	Break at 0. Goes to $+\infty$ and $-\infty$
$f(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ 2 & \text{if } x > 0 \end{cases}$	Left limit = 1, Right limit = 2. Left limit and Right limit exist but are not equal. So limit at 0 doesn't exist.	Gap at 0
$f(x) = \begin{cases} x-2 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x+2 & \text{if } x > 0 \end{cases}$	Left limit = -2, Right limit = 2. Left limit and Right limit exist but are not equal. So limit at 0 doesn't exist.	Gap at 0
$f(x) = \begin{cases} x^2-1 & \text{if } x \leq 1 \\ -x^2-1 & \text{if } x > 1 \end{cases}$	Left limit = 0, Right limit =-2. Left limit and Right limit exist but are not equal. So limit at 0 doesn't exist.	Gap at 0

Activity 13.4 Limit of Trigonometric Functions

Observes the the graph of $\sin x$ and x by zooming at the origin	While zooming at the origin graph of $\sin x$ seems to coincide with the graph of x. That is as $x \to 0$ $\sin x$ becomes 'almost equal' to x. So $\lim_{x\to 0} \frac{\sin x}{x} = 1$ [1]	Gets the limit $\lim_{x\to 0} \frac{\sin x}{x} = 1$ geometrically.
Observes the graphs of the functions x^2, $\sin x^2$, $\sin^2 x$, $\tan x^2$ and $\tan^2 x$ by zooming at the origin.	Graphs of all of the functions seems to coincide at the origin. That is as $x \to 0$ values of above functions becomes 'almost equal'. This implies that the limit of the functions of the form $\frac{f(x)}{g(x)}$, where f and g are any of the functions given $(\frac{\sin x^2}{x^2}, \frac{\sin^2 x}{\tan x^2}, etc.)$, as $x \to 0$ is 1	Derives the value of the limits $\lim_{x\to 0} \frac{\sin x^2}{x^2}$, $\lim_{x\to 0} \frac{\sin^2 x}{\tan x^2}$, etc. geometrically.

1. If the graphs of two functions f and g meets tangentially at some point then most probably the limit of the function $\frac{f(x)}{g(x)}$ at that point is 1. We use this to derive some limits geometrically.

Activity 13.5 Limit of Exponential and Logarithmic Functions

☛ Creates the applet. ☛ Finds the equation of the curves when they touch the line $y = x$. ☛ Observes the curves by zooming at the origin.	☛ Recall the concept of shifting of curves. ☛ Ask the students to imagine the shift of the curves so that they meet tangential to the line $y = x$ at the origin and to find the corresponding equations. ☛ Definition of the functions become $e^x - 1$ and $\log(1 + x)$.By zooming at the origin, their graphs seems to coincide with the line $y = x$. This implies the limits $\lim_{x\to 0} \frac{e^x-1}{x} = 1$ and $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$ ☛ If both left and right limits exist and are equal then the limit exist at that point	Derives the limits $\lim_{x\to 0} \frac{e^x-1}{x} = 1$ and $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$ geometrically.

Activity 13.A Some more problems

Observes the nature of the function $f(x) = \sin\frac{1}{x}$ as $x \to 0$	As $x \to 0$ $\sin\frac{1}{x}$ oscillates between 1 and -1 hence $\lim\limits_{x\to 0}\sin\frac{1}{x}$ doesn't exist.	Realise that $\lim\limits_{x\to 0}\sin(\frac{1}{x})$ doesn't exist
Observes the nature of the function $f(x) = x\sin\frac{1}{x}$ as $x \to 0$	☛ As $x \to 0$, even though the function $x\sin\frac{1}{x}$ oscillates, it oscillates between x and $-x$ (discuss the reason [1]). x and $-x$ approaches 0 as $x \to 0$. Hence $\lim\limits_{x\to 0} x\sin\frac{1}{x} = 0$. ☛ Sandwich theorem may be discussed ☛ The following table may be used for the discussion [2]	Realise that $\lim\limits_{x\to 0} x\sin\frac{1}{x} = 0$
Observes the nature of the function $f(x) = x^2\sin\frac{1}{x}$ as $x \to 0$	As $x \to 0$, even though the function $x^2\sin\frac{1}{x}$ oscillates, it oscillates between x^2 and $-x^2$. x^2 and $-x^2$ approaches 0 as $x \to 0$. Hence $\lim\limits_{x\to 0} x^2\sin\frac{1}{x} = 0$.	Realise that $\lim\limits_{x\to 0} x^2\sin\frac{1}{x} = 0$
Observes the nature of the function $f(x) = \sqrt{x}\sin\frac{1}{x}$ as $x \to 0$	As $x \to 0$, even though the function $\sqrt{x}\sin\frac{1}{x}$ oscillates, it oscillates between $\sqrt{x}$ and $-\sqrt{x}$. We can see that the graph of the function $\sqrt{x}\sin\frac{1}{x}$ oscillates inside the parabola $x = y^2$. Hence $\lim\limits_{x\to 0}\sqrt{x}\sin\frac{1}{x} = 0$.	Realise that $\lim\limits_{x\to 0}\sqrt{x}\sin\frac{1}{x} = 0$

1. $\sin x$ lies between -1 and 1. Hence $\sin\frac{1}{x}$ also lies between -1 and 1.

$$-1 \leq \sin\frac{1}{x} \leq 1$$
$$\Rightarrow -x \leq x\sin\frac{1}{x} \leq x$$

So the graph of $x\sin\frac{1}{x}$ lies between the graphs of $-x$ and x

2. From the following table we can see that as $x \to 0$ $\sin\frac{1}{x}$ oscillates between 1 and -1, while $x\sin\frac{1}{x}$, $x^2\sin\frac{1}{x}$ and $\sqrt{x}\sin\frac{1}{x}$ approaches 0.

x	$\frac{1}{x}$	$\sin\frac{1}{x}$	$x\sin\frac{1}{x}$	$x^2\sin\frac{1}{x}$	$\sqrt{x}\sin\frac{1}{x}$
$\frac{2}{\pi}$	$\frac{\pi}{2}$	1	$\frac{2}{\pi}$	$(\frac{2}{\pi})^2$	$\sqrt{\frac{2}{\pi}}$
$\frac{2}{2\pi}$	$\frac{2\pi}{2}$	0	0	0	0
$\frac{2}{3\pi}$	$\frac{3\pi}{2}$	-1	$-\frac{2}{3\pi}$	$(-\frac{2}{3\pi})^2$	$-\sqrt{\frac{2}{3\pi}}$
$\frac{2}{4\pi}$	$\frac{4\pi}{2}$	0	0	0	0
$\frac{2}{5\pi}$	$\frac{5\pi}{2}$	1	$\frac{2}{5\pi}$	$(\frac{2}{5\pi})^2$	$\sqrt{\frac{2}{5\pi}}$
.	.	.	.	.	.
.	.	.	.	.	.

Lab 14

Derivative at a point

In this lab we have three activities through which we try to establish the geometric meaning of derivative at a point.

Required Concepts

- ☛ To identify a tangent line in terms of secant lines.
- ☛ To explore the geometrical interpretation of the derivative of a function at a point.
- ☛ To explore different cases of non-differentiability of a function at a point.

Aim
Here we find the slope of a tangent line to a curve as the limiting case of the slope of a secant line. Also discuss the case where this limit (derivative) does not exist and its geometrical interpretation.

Activity 14.1 Geometrical Meaning of Derivative at a Point

This activity is done using the GeoGebra applet ML 14.1. Students familiarises the initial settings in the applet as described in the Lab Manuel.

Changes the value of h from 1 to 0 and from -1 to 0. Observes the value of slope of the secant line.	☛ What happens to the secant line as h $\Rightarrow$ 0 ? ☛ Why the slope of the secant line vanishes when h = 0? ☛ How can we find the slope of the secant line when $h \neq 0$? ☛ How can we find the slope of the tangent line ?(Discuss the method)[1] ☛ Discuss the definition of tangent[2]	Realises the following ☛ As h $\Rightarrow$ 0, secant line approaches the tangent line at P ☛ When h = 0, Q coincides with P, so secant line is not defined. Hence it vanishes.So it is not possible to find the slope of the tangent directly. ☛ Slope of the tangent is the limiting case of the slope of the secants as h $\Rightarrow$ 0

1. Slope of the tangent line is the limit of slope of secant as h $\Rightarrow$ 0. That is $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$. This limit, if it exists, is called the derivative of $f(x)$ at $x = a$ and is denoted by $f'(a)$.

2. Students are already familiar with tangent to a circle , which is defined as a line passing through a point on the circle and perpendicular to the radius of the circle at that point. But it is not possible to define tangent to a curve in general without using the concept of limits. Definition of tangent to a curve is as follows.

Tangent to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line passing through P and having slope $f'(a)$, if it exists.

Activity 14.2 Derivative at a Point

Edit the function and the point using input boxes for f and a and finds the slope of tangent	☛ Ask the students to draw the curves and tangents at the given points and to verify their answers using 'Slope' tool. ☛ Give examples to show that tangent to a curve may cut the curve or touch the curve in more than one point [1]	Gets the slope of the tangents

1. Students may have a miss-concept that tangent to a curve just touches the curve in a single point without cutting the curve. Give examples to clarify this.

   In example 4, tangent to the curve $y = x^3$ at $(0, 0)$ is the x axis, which cuts the curve
   In example 5, tangent to the curve $y = \sin x$ at $(0, 0)$ is the line $y = x$, this also cuts the curve
   In example 6, tangent to the curve $y = \sin x$ at $x = \frac{\pi}{2}$ is the line $y = 1$, this touches the curve at $x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots$

   Note:-
   In some examples it may see a slight difference between the limiting values of the slope of the secant lines as h $\Rightarrow$ 0 from right and left. In such situations we can change the increment of the slider **h** to 0.001 or 0.0001 ,... (Right click on **h** $\Rightarrow$ Object Properties $\Rightarrow$ Slider $\Rightarrow$ Increment) and increase the number of decimal places to be shown (Options $\Rightarrow$ Rounding $\Rightarrow$ select 3 or 4 , ...)

Activity 14.3 Non Differentiability - Geometrical Meaning

Observes the secant line and its slope as h $\rightarrow$ from left and right.	☛ Discuss different cases of non existence of derivative ☛ Discuss the nature of the curve at a point where the function is differentiable/non differentiable	Familiarise with different cases of non differentiability

Analysis of the Problems

Function	Existence/Non existence of Derivative	Nature of curve
$f(x) = \|\sin x\|$ at $x = 0$	As $h \to 0$ from the right secant line approaches the line $y = x$. So $\lim\limits_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = 1$. As $h \to 0^-$ secant line approaches the line $y = -x$. So $\lim\limits_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = -1$. So $\lim\limits_{h\to 0} \frac{f(0+h)-f(0)}{h}$ doesn't exist. That is $f'(0)$ doesn't exist. Hence tangent doesn't exist at $(0, 0)$.[1]	Curve is sharp at $x = 0$
$y = \sin x$ at $x = \pi$	Left derivative = Right derivative = -1. So $f'(\pi) = -1$. Secant lines from both sides approaches to $y = -x+\pi$. Tangent is $y = -x+\pi$	Smooth at $x = \pi$
$y = \sin \|x\|$ at $x = \pi$	Left derivative = Right derivative = -1. So $f'(\pi) = -1$. Secant lines from both sides approaches to $y = -x+\pi$. Tangent is $y = -x+\pi$	Smooth at $x = \pi$
$y = \|x\|$ at $x = 0$	Left derivative = -1, Right derivative = 1. So $f'(0)$ doesn't exist. Secant lines approaches to different lines.Tangent doesn't exist	Sharp at $x = 0$
$y = \begin{cases} x^2 & \text{if } x \leq 2 \\ 2x & \text{if } x > 2 \end{cases}$ at $x = 2$	Left derivative = 4, Right derivative = 2. $f'(2)$ doesn't exist. Secant lines approaches to different lines. Tangent doesn't exist	Sharp at $x = 2$
$y = x^{\frac{1}{3}}$ at $x = 0$	Secant lines from both sides approaches y axis. Hence the tangent exist (y axis) but since the slope of y axis is not defined, function is not differentiable at $x = 0$	Smooth at $x = 0$ [2]
$y = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x \neq 2 \\ 6 & \text{if } x = 2 \end{cases}$ at $x = 2$	Secant lines from both sides approaches a vertical line. Hence derivative doesn't exist at $x = 2$	Break at $x = 2$ [3]
$y = \begin{cases} x^2 & \text{if } x \leq 2 \\ (x-4)^2 & \text{if } x > 2 \end{cases}$ at $x = 2$	Left derivative = 4, Right derivative = -4. $f'(2)$ doesn't exist. Secant lines approaches to different lines. Tangent doesn't exist	Sharp at $x = 2$
$y = \begin{cases} x^2 & \text{if } x \leq 2 \\ (x-2)^2 & \text{if } x > 2 \end{cases}$ at $x = 2$	From the left side, secant line approaches to a line having slope 4 and from the right, it approaches to a vertical line. Hence the left derivative is 4 and right derivative doesn't exist. Tangent doesn't exist	Curve has a break at $x = 2$

1. Since $\lim\limits_{h \to 0^+} \frac{f(0+h)-f(0)}{h} = 1$ we may call it as right derivative and the line $y = x$ as the right tangent. Similarly $\lim\limits_{h \to 0^-} \frac{f(0+h)-f(0)}{h} = -1$ is the left derivative and the line $y = -x$ is the left tangent.
2. Differentiability is an indication of the smoothness of the curve. That is, if a function is differentiable at a point the corresponding curve is smooth at that point. But the converse of this is not true. $x^{\frac{1}{3}}$ is not differentiable at 0, but $y = x^{\frac{1}{3}}$ is a smooth curve.
3. Break is not visible from the graph. We can convince the students as follows. Create a slider a and plot the point $(a, f(a))$. As the slider changes we can see that the point moves along the line, jumps to $(2, 6)$ when a becomes 2 and comes back to the line when a changes.

Lab 15

Derivative of a function

Derivative of a function at a point and its geometrical meaning is already discussed in Lab14. In this lab we discuss derivative of a function, its meaning, graph of derived function. In the additional activity we discuss 'Function Machine'.

Required concepts

- Concept of the derivative of a function at a point.
- Derivative of a function at a point is the slope of the tangent to the corresponding curve at that point.
- Graph of a function and its derivative.

Aim

- Students get a clear idea about the derivative of a function.
- Establishes a relation between a function and its derivative.

Activity 15.1 Relation Between a Function and its Derivative

Creates the applet and finds the slope for different values of a.	• Recall the concept of derivative at a point • Recall the meaning of derivative at a point, that is the slope of the tangent at that point.	• Completes the table • Identifies that $f'(a) = 2a$

Activity 15.2 Graph of Derived Function

Observes the path of the point $C(a, m)$	• Recall the concept of graph of a function • Discuss the concept of derived function [1] • Discuss the methods of finding the equation of the path [2]	• Gets a clear idea about the derivative of a function and its graph. • Gets the equation of the path as $y = 2x$

1. C is the point $(a, f'(a))$. So for a given x, the corresponding point on the curve traced by C is $(x, f'(x))$. Hence the curve represents the function $y = f'(x)$, the derivative or derived function of $f(x)$.
2. ☛ The path traced by the point C is straight line. So by observation we can find its equation as $y = 2x$
   ☛ In Activity 15.1 we found the relation between a and $f'(a)$ as $f'(a) = 2a$. That is the y coordinate of any point on the curve is twice the x coordinate. Hence the equation is $y = 2x$.

Activity 15.3 Equation of Derived Function

Observes the path of the point $C(a, m)$ for different functions.	☛ Discuss the methods of finding the equation of the path [1] ☛ Discuss the fact that adding a constant doesn't changes the derivative.	☛ Gets a clear idea about the derivative of a function and its graph. ☛ Gets the equation of the path for the given functions ☛ Completes the table ☛ Understands that the equation of the path gives the derivative of the function.

1. ☛ Function $x^2 + 1$: Since the curve is a straight line passing through the origin its equation is $y = kx$. It passes through the point $(1, 2)$. So $k = 2$ and the equation is $y = 2x$
   ☛ Function $5x^2$: Here also the curve is a straight line passing through the origin and through the point $(1, 10)$. The equation is $y = 10x$
   ☛ Function x^3 : Curve is a parabola of the form $x^2 = 4ay$. We re write it in the form $y = kx^2$. It passes through the point $(1, 3)$. So $k = 3$ and the equation is $y = 3x^2$
   ☛ Function $x^3 - 2$: Same as above
   ☛ Function $\sin x$ By observation it is easy to identify that the curve represents $\cos x$
   ☛ Function $\cos x$ We can see that the curve is the reflection of $\sin x$ on x axis. Hence it is $-\sin x$.
   ☛ Give more functions such as x, $3x + 2$, $f(x) = 3$, etc. as an exercise.

Activity 15.4 Derivative Using Command

Finds the derivative of the functions using input commands	☛ If graph of the derivative appears on Graphics view, we can shift it to Graphics 2 as follows. Right click on the graph → Object Properties → Advanced → Graphics 2 → uncheck Graphics. ☛ Note [1]	☛ Gets the derivative of the functions ☛ Completes the table

1. Derivative of some functions may appear differently from that given in the text book. For example geoGebra shows the derivative of $\tan x$ as $1 + \tan^2 x$ which is same as $\sec^2 x$

Activity 15.A Derivative Machine

In Lab 1, Activity 1.3 we have compared a function with a machine which gives an output, according to the definition of the function, for a given input. Derivative of a function is again a function. So in this activity we treat "Derivative" as an operator, which changes one function to another. So we compare it with an input output machine. Here the inputs are functions. Since functions can be treated as machines, we consider Derivative Machine as a super machine which transforms a machine into another.

Lab 16

Miscellaneous

This lab consists of four activities and three additional activities. The activities deals with concepts from Complex numbers, Sequence and Series and Linear Inequalities. Additional activities deals with the geometry of sum, product and square root of complex numbers.

Activity 16.1 Complex numbers

Required concepts

- ☛ Modulus and argument of a complex number.
- ☛ Representation of a complex number on the Argand plane.
- ☛ Polar form of a complex number

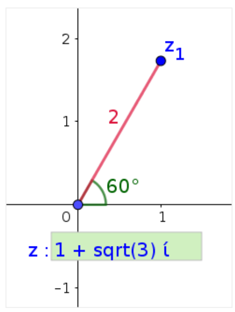
Aim

- ☛ To explore the geometry of modulus , argument and polar form of a complex number.

Creates the applet and finds the modulus and argument of the complex numbers geometrically.	☛ Discuss the method of finding modulus and argument geometrically.[1] ☛ Discuss the method of finding the polar form.[2] ☛ Discuss the method of plotting complex numbers whose polar coordinates are given [3]	☛ Gets the modulus and argument of the given complex numbers. ☛ Finds the polar form.

1. Plot the given complex number and the point $(0,0)$. Draw the line segment joining the complex number with the origin. Find its length and find the angle made by the line segment with the positive direction of the x axis (use the tools 'Distance or Length, and 'Angle') Length of the line segment gives the modulus and the angle gives the argument.

2. Using the modulus and argument obtained from the figure, we can write the polar form of the complex number



3. To plot the complex number $(2, 40°)$ give the input command $2(\cos(40°) + i * \sin(40°))$

Activity 16.2 Sequence and Series

Required concepts

- General term of a sequence.

Aim

- Through this activity we generate sequences using GeoGebra commands which helps the students to get a clear idea about the general term of a sequence.

Finds the general term of the given sequences and generates them using GeoGebra	Discuss the method of finding general term. *	Generates the sequences

*

Sequence	General Term	Command
6, 10, 14...	$4n+2$	`Sequence(4n+2,n,1,m)`
2, 4, 8...	2^n	`Sequence(2^n,n,1,m)`
$1, \frac{1}{2}, \frac{1}{4}...$	$\frac{1}{2^n}$	`Sequence(1/2^n,n,0,m)` [1]
0.1, 0.01, 0.001...	$\frac{1}{10^n}$	`Sequence(1/10^n,n,0,m)`
$\frac{1}{9}, \frac{-1}{27}, \frac{1}{81}, \frac{-1}{243}...$	$(-1)^n \frac{1}{3^{n+1}}$	`Sequence((-1)^n*1/(3^(n+1)),n,1,m)`

1
This command gives the terms of the sequence in decimals. We can show the terms as fractions as follows.
Create a list (say L_1) using the command `{Sequence(2^n, n, 0, m)}` Create a list (say L_2) using the command `{Sequence("\frac{1}{{" Element(L_1, i) "}}  ,", i, 1, m + 1)}`
Create the list using the command `{TableText({L_2})}`

Activity 16.3 Sum to n terms

Required concepts

- General term of a sequence.

Aim

- Uses GeoGebra commands to find the sum to a required number of terms of sequences and series. We also discuss the sum to infinity of a Geometric Progression.

☛ Finds the general term and sum to required number of terms. ☛ Observes the sum as the number of terms increases	Discuss the convergence of sum of Geometric progression.	☛ Finds the sum of given sequences. ☛ Gets a clear idea about the sum to infinity of a Geometric progression.

Activity 16.4 Graphical Solution of Linear Inequalities

Students' Activity	Teachers' role	Students' Response/Findings
Draws the regions representing the given inequalities	Discuss the logic behind the input command. [1]	Finds the solution of the system of linear inequalities.

1. Consider the commands

   x+2y¡=8&&2x+y¡=8&&x¿=0&&y¿=0

   OR

   x+2y $\leq$ 8 $\wedge$ 2x+y $\leq$ 8 $\wedge$ x $\geq$ 0 $\wedge$ y $\geq$ 0

   In this commands we are combining the following inequalities using 'and'.

   $x + 2y \leq 8$
   $2x + y \leq 8$
   $x \geq 0$, $y \geq 0$

   Note that we use && or $\wedge$ for connecting the inequalities. $\wedge$ is available from the drop down menu given on the right end of the input bar.

Activity 16.A Sum of Complex Numbers

☛ The quadrilateral obtained is a parallelogram. Discuss the method of finding sum of two complex numbers geometrically by completing the parallelogram. We may relate this with addition of vectors studied in physics class.

Activity 16.B Product of Complex Numbers

☛ If r_1, r_2 are the modulus and θ_1, θ_2 are the argument of the complex numbers then the modulus of their product is $r_1 r_2$ and argument of the product is $\theta_1 \theta_2$

☛ Multiplication by i rotates the complex number by an angle 90° in the anti-clockwise direction.

☛ Multiplication by $-i$ rotates the complex number by an angle 90° in the clockwise direction.

☛ Modulus of $\frac{z_1}{z_2}$ is $\frac{r_1}{r_2}$ and its argument is $\theta_1 - \theta_2$

Activity 16.C Square Root of a Complex Number

- Argument of $\sqrt{z_1}$ is the half of argument of z_1
- Modulus of $\sqrt{z_1}$ is the square root of the modulus of z_1
- De Moivre's theorem may be discussed

If $z = r(\cos(\theta) + i\sin(\theta))$ then $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$

Appendices

Appendix A

Statistics using LibreOffice Calc

LibreOffice Calc is a software from the LibreOffice suite of applications predominantly dealing with data analysis, where data is represented in the form of tables. Teaching of statistics can be made more pleasing with this application as the following sections will show. Here we will brush through the concepts presented in the statistics chapter from the plus one textbook. The emphasis will be on discovery and self learning.

Activity A.1 Mean and Median

We have obtained the scores of two batsmen in different matches from media reports. Which batsman is better? How can we compare the performance of the two batsmen? The scores of the two batsmen A and B in the last ten matches is given.
Batsman A: 30,91,0,64,42,80,30,5,117,71
Batsman B: 53,46,48,50,53,53,58,60,57,52
We will take the help of LibreOffice Calc to explore the differences between the two batsmen.

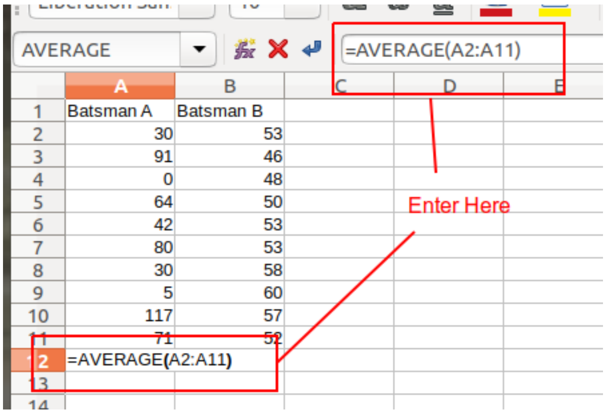
Open LibreOffice Calc. Data is handled in rows and columns in Calc. Data can be entered into the application either row wise or column wise. We will use the first column for Batsman A and the second column for Batsman B. The entries in each row (we call them as cells) would represent the runs obtained in a particular match. A particular cell is identified by its row number and column number. If you see on the top of the columns it will be marked A,B,C etc. which represents the column and in the left end of each row you can find the row number. So A2 represents the first column and second row.

Enter the data given as shown in Figure A.1. The data in the figure for Batsman A is entered from the cell A2 to A11 and for Batsman B it is entered from the cell B2 to B11.

Let us now calculate the mean of this data. We will display the mean of Batsman A in the cell A12 and mean of Batsman B in the cell B12. To obtain the mean we use a Calc function called

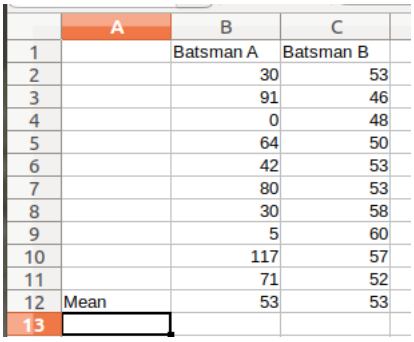
	A	B
1	Batsman A	Batsman B
2	30	53
3	91	46
4	0	48
5	64	50
6	42	53
7	80	53
8	30	58
9	5	60
10	117	57
11	71	52
12		

Figure A.1: Initial Data Entry



	A	B
1	Batsman A	Batsman B
2	30	53
3	91	46
4	0	48
5	64	50
6	42	53
7	80	53
8	30	58
9	5	60
10	117	57
11	71	52
12	=AVERAGE(A2:A11)	

Figure A.2: To find the mean



	A	B	C
1		Batsman A	Batsman B
2		30	53
3		91	46
4		0	48
5		64	50
6		42	53
7		80	53
8		30	58
9		5	60
10		117	57
11		71	52
12	Mean	53	53
13			

Figure A.3: The mean of both players

AVERAGE. Click on the cell A12 and type "=AVERAGE(A2:A11)" as shown in Figure A.2 (note that the data we entered for Batsman A is from A2 to A11). This can also be entered in the input box in the top of the tool area as shown in Figure A.2. After entering the data press the enter key to obtain the value of the mean. Repeat the same process for Batsman B where the formula entered would be "=AVERAGE(B2:B11)".

We can see that the mean of both the players is the same. So we cannot infer the difference between the players from the mean data.

To make more sense out of our data we will insert a column before the first column. To do this, select the column A by clicking on it. Right click and select "Insert columns to the left". Now our data will be in columns B and C and in the cell A12 type "Mean" as shown in Figure A.3.

Now let us check the median value to see if we can draw any conclusion about the performance of the batsmen. For finding the median which is the central value of the data, we first sort the available data. Select the cells from B2 to B12. In the menu "Data" select the option "Sort Ascending". Click on "Current Selection". We can see that the data of column B is now sorted in the ascending order. Repeat this same process to column C as well to obtain the sorted data of Batsman B (Figure A.4). Since there are ten sets of data here, the central value will be the average of 5th and 6th values. For Batsman A, the average of 42 and 64 is to be taken (which is 53) and for Batsman B it is 53.

The median can also be obtained from the unsorted data using the MEDIAN function. Click on the cell B13 and type "=MEDIAN(B2:B11)" as shown in Figure A.5. On click of the enter key, the value of median is computed. Repeat the same for the data of second batsman as well.

	A	B	C
1		Batsman A	Batsman B
2		0	46
3		5	48
4		30	50
5		30	52
6		42	53
7		64	53
8		71	53
9		80	57
10		91	58
11		117	60
12	Mean	53	53

Figure A.4: The median of both players

AVERAGE =MEDIAN(B2:B11)

	A	B	C	D
1		Batsman A	Batsman B	
2		30	53	
3		91	46	
4		0	48	
5		64	50	
6		42	53	
7		80	53	
8		30	58	
9		5	60	
10		117	57	
11		71	52	
12	Mean	53	53	
13		=MEDIAN(B2:B11)		
14				
15				

Figure A.5: The median of both players

Activity A.2 Measures of Dispersion of Data

The values of mean and median we saw in the above section has failed to convey the difference between the scores of the two batsmen. Let us explore the data provided further. We shall attempt to plot this data as a line graph to explore whether it can throw some light into the differences in the scores.

Select the two columns containing data (in our case it is the columns B and C) from B2 to C11. In the menu select the option Insert and then chose Chart. In the chart popup that comes up, select the line chart with the option to include lines and points. When the Finish button is clicked, the chart as shown in Figure A.6 comes up.

The figure clearly shows that for Batsman A, the data is more spread from very small scores to very large scores whereas for Batsman B, the scores are more or less crowded around the mean value. So to obtain a measure of this tendency to deviate from the mean value, we look at the differences from the central value (mean).

To check this we will compute the differences of individual data from the mean value, with a hope of obtaining some insight. We will compute the differences in the column D. We will make use of formulas to compute the differences. So click in the cell D2 and enter "=B2-B$12". Here B2 has the first score and B12 has the mean value. Note the $ sign before 12. We intend to say that 12 is an unchanging value - something like a constant, whereas the 2 in B2 does not have the $ sign indicating that it is a value that is liable to change. So for the next match the same formula if copied down will automatically become B3-B12. Once enter key is pressed the difference from the mean is computed.

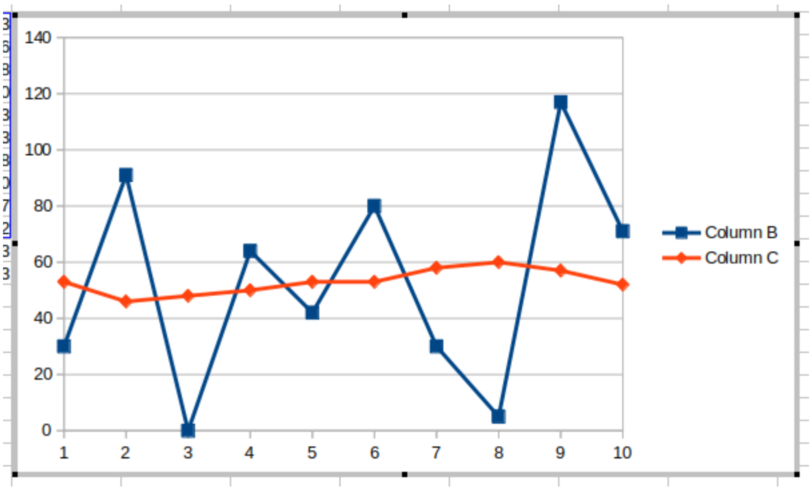


Figure A.6: The spread of scores

	A	B	C	D	E
1		Batsman A	Batsman B	Deviations from mean	
2		30	53	-23	0
3		91	46	38	-7
4		0	48	-53	-5
5		64	50	11	-3
6		42	53	-11	0
7		80	53	27	0
8		30	58	-23	5
9		5	60	-48	7
10		117	57	64	4
11		71	52	18	-1
12	Mean	53	53	0	0
13	Median	53	53		
14					

Figure A.7: The deviations from mean and its sum

Copy this cell (D2) and paste in all the cells D3 to D11. All the differences from mean value is displayed. Copy the cells from D2 to D11 and paste in the cells E2 to E11. So we have the deviations from the mean for each game of Batsman A in column D and that for Batsman B in column E as shown in Figure A.7. Click on cell D1 and enter the text "Deviations from mean" so that it is easy to distinguish between the data later on.

Now let us calculate the sum of these deviations. Click on the cell D12 and enter the formula to compute the sum as "=SUM(D2:D11)". Press the enter key to display the sum. Similarly repeat the same for the column E by using the formula "=SUM(E2:E11)" in Cell E12.

We can see that the sum of the deviations from mean is zero for both the cases. Why is it so? We had computed the sum of deviations from the mean value which is

$$x_1 - \bar{x} + x_2 - \bar{x} + \cdots + x_n - \bar{x} = x_1 + x_2 + \cdots + x_n - n\bar{x} = \sum_{j=1}^{n} x_j - \sum_{j=1}^{n} x_j = 0$$

So this result is expected. But still we have not come up with a measure to bring out the differences in the score. The sum of deviations turned out to be zero because there were some positive numbers and there were some negative numbers. What if we take the sum of absolute values? Let us check. For this we simply change the formula employed in cell D2 to "=ABS(B2-B$12)". ABS is the function that would calculate the absolute value. Do the procedure detailed as above for all the other cells as well. For D12 use the "=AVERAGE(D2:D11)" to find the mean of these absolute values. Repeat the same for the cell E12 as well as shown in the Figure A.8

	A	B	C	D	E
1		Batsman A	Batsman B	Deviations from mean	
2		30	53	23	0
3		91	46	38	7
4		0	48	53	5
5		64	50	11	3
6		42	53	11	0
7		80	53	27	0
8		30	58	23	5
9		5	60	48	7
10		117	57	64	4
11		71	52	18	1
12	Mean	53	53	31.6	3.2
13	Median	53	53		
14					

Figure A.8: The absolute deviations from mean and its mean

	A	B	C	D	E	F	G	H
1		Batsman A	Batsman B	Deviations from mean		Square of deviations from mean		
2		30	53	23	0	529	0	
3		91	46	38	7	1444	49	
4		0	48	53	5	2809	25	
5		64	50	11	3	121	9	
6		42	53	11	0	121	0	
7		80	53	27	0	729	0	
8		30	58	23	5	529	25	
9		5	60	48	7	2304	49	
10		117	57	64	4	4096	16	
11		71	52	18	1	324	1	
12	Mean	53	53	31.6	3.2	1300.6	17.4	
13	Median	53	53					
14								

Figure A.9: The varience

The values obtained for both the players is seen to be different. For the Batsman A, we see that the value is 31.6 while that of Batsman B it is 3.2. The small difference of this value with the mean value shows that the performance of Batsman B is more consistent than that of Batsman A.

Activity A.3 Varience

To get the deviation from men value, we had tried to take the absolute value of all the differences to get rid of the negative values. Another approach which is more consistent would be to take the square of all the values. This would automatically get rid of the negatives.

To compute this value we will again make use of formulas. In the cell F2 we will enter the formula "=(B2-B$12)2̂" which is the square of the differences from the mean value. Press on the enter key to evaluate the square of the deviation. Copy the cell F2 and paste in F3 to F11. Copy cells from F2 to F11 and paste in cells G2 to G11.

Now we will compute the mean value of this in cells F12 and G12. For Batsman A we can see that it is 1300.6 and that of Batsman B it is 17.4 as shown in Figure A.9. So this measure also brings out the differences between the two batsmen. This measure is termed as the varience.

Mathematically, varience can be calculated by the equation

$$\sigma^2 = \frac{1}{n}\sum_{j=1}^{n}(x_j - \bar{x})^2$$

Activity A.4 Standard Deviation

In the above section we computed the varience which is the square of the deviations from the mean value. It is the square of the data that we are getting rather than the data. So it would be more realistic to take the square root of this value to get a value that represents the differences.

We use formulas to do it. In the cell F13 type the formula "=SQRT(F12)". SQRT is the formula to calculate the square root. This value is called the standard deviation. For Batsman A it is about 36 and for Batsman B it is about 4. The larger the standard deviation, the more spread is the data from the central value.

Mathematically, the equation for standard deviation is

$$\sigma = \sqrt{\frac{1}{n}\sum_{j=1}^{n}(x_j - \bar{x})^2}$$

The standard deviation can also be calculated directly from the given sample using the formula "=STDEV.P(B2:B11)" where STDEV is the formula for standard deviation and the .P in the formula specifies that we are considering the whole population for computing the standard deviation.

Appendix B

Using Python in Mathematics Instruction

Python is a popular programming laguage which is simple compared to other programming languages. Its strength can be explored in the classroom as a teaching aid and can be used by the students and teachers to explore various fascinating aspects of mathematics.

Many books have been written that deal with Python as a programming language and there are a number of online resources to learn Python. Teaching or learning Python is not an objective of this chapter. The aim of this chapter is to show that Python can be used in the classroom as an effective aid to the teaching learning process.

Activity B.1 Installing and opening Python

In linux distributions, Python will be installed by default. To check whether Python is installed or not, open Terminal (Applications → Accessories → Terminal). A screen which is normally called as console opens up. Type python in the console. If python is installed then the version of python will be shown in the screen as shown in Figure B.1

Python programming can be done in this console itself. But for convenience, we normally use an editor for the programming jobs. An editor which is popular among the python community is IDLE. To install IDLE in your system type the following commands in succession (Please note that the system should be connected to the internet while performing this action.)

```
sudo apt-get update
sudo apt-get install idle
```

After the first command, the system will prompt for the password. Once IDLE is installed it can be opened by typing idle in the terminal. A screen opens up with the idle interface as shown in Figure B.2.

Activity B.2 Starting Programming

To start with, let us type the following command in the idle window

```
print("Hello Python")
```

When the enter key is pressed, the output is visible in the screen as shown in Figure B.3.

Python can be used to do calculation as the following examples show. Enter the command below

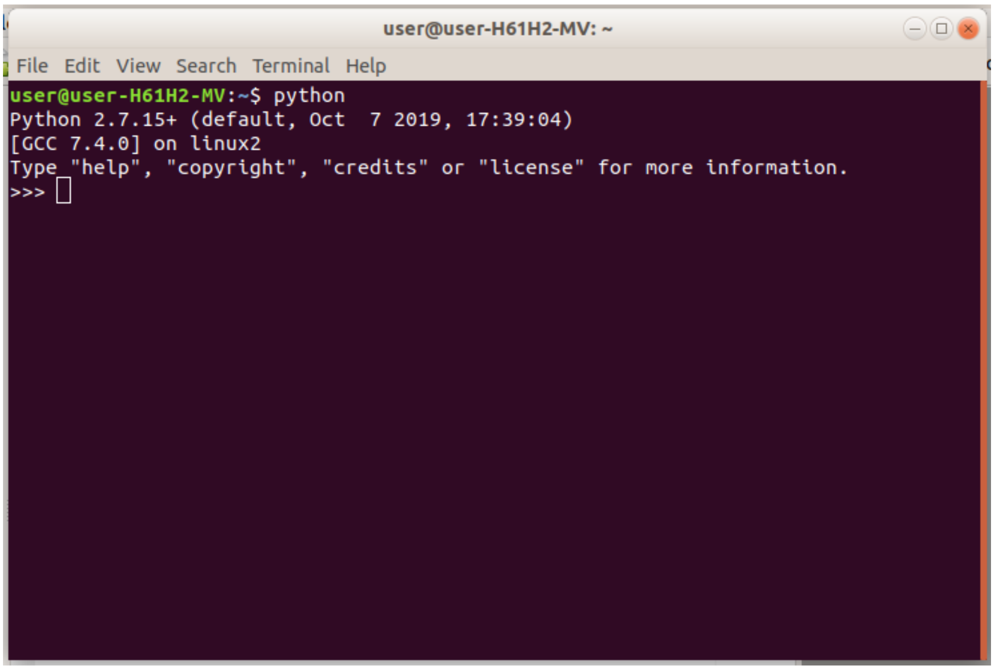


Figure B.1: Checking whether Python is installed

```
2*7+11*3
```

to get the answer as 47. Here * is for multiplication and + is for addition. To compute exponents we use ** as in the code below which computes $2^{1}0$

```
2**10
```

Division is performed using / as below

```
6/4
```

which gives 1.5 while the following code

```
6//4
```

gives 1. In this case the double division operator calculates the integer value of the division.

Python can handle arbitrarily large numbers. For example, try to compute 2^{1000} as in

```
2**1000
```

to get the result shown in Figure B.4

Variables can also be used for computations like in

```
a=2.0
u=3.0
t=10
s=u*t+(1/2)*a*t**2
```

To get the value of the variable s type the name of variable in the idle screen and press enter.

There are many built in functions in Python that could be helpful in mathematical calculations. These functions are defined in a class file called "math" which has to be imported to the Python environment. For this the following commands can be tried out.

```
from math import *
exp(1)
sqrt(2)
```

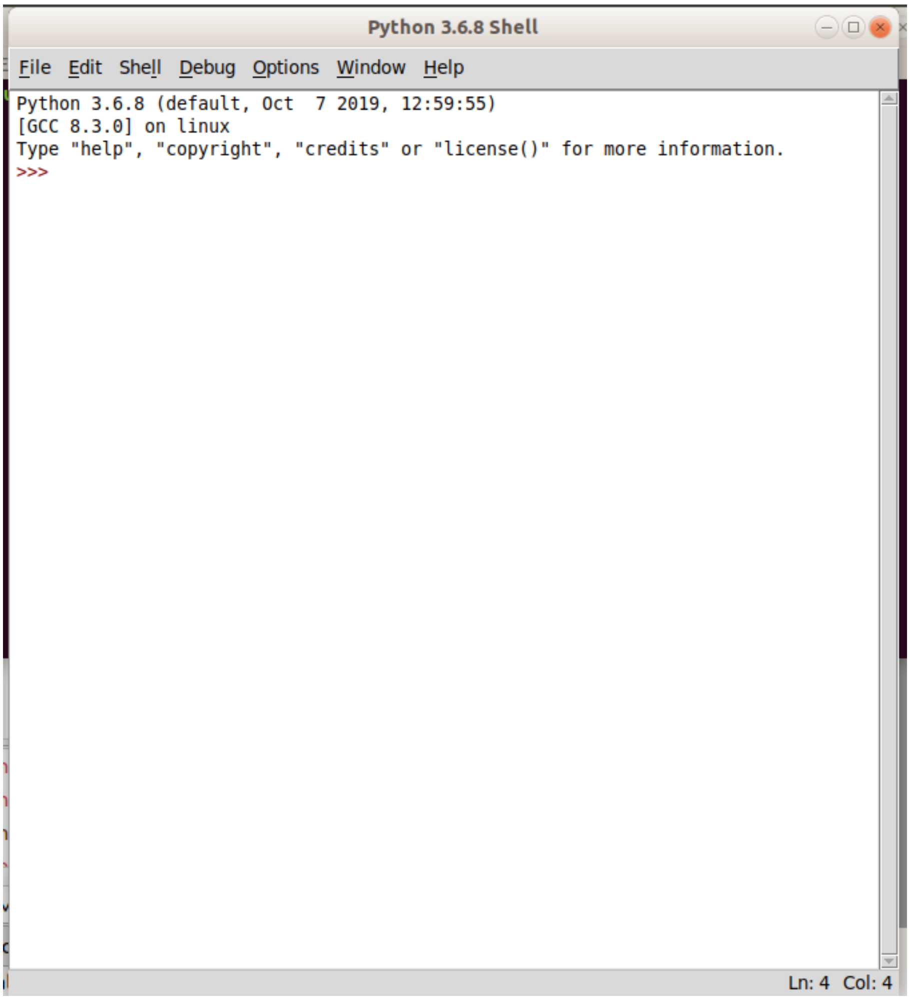


Figure B.2: IDLE interface

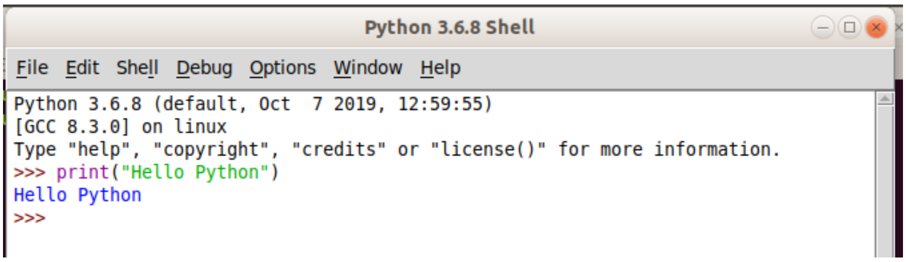


Figure B.3: The first program

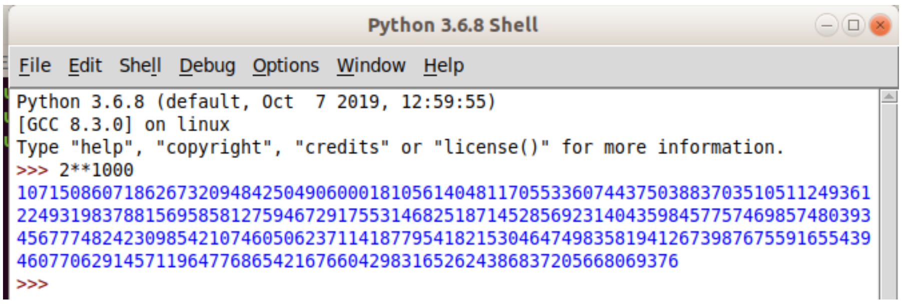


Figure B.4: Handling large numbers

Here * symbol in the import statement asks to import all the functions defined in "math". The first statement calculates the value of the natural constant e and the second statement calculates the value of $\sqrt{2}$

Suppose we want to find the sum of the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{100}$$

To do this we need to first store our result in a variable which we will call as sum. We then make the value of this variable as zero. Now we start from the first term and compute the value of the first term. We keep on adding this value to the variable sum until we reach the end of the series. The Python code for this logic is shown below.

```
sum=0
for k in range(1,101):
    sum = sum + 1.0/k
sum
```

Here you can see that the answer is the sum of the first hundred terms of the series. Now let us try to understand the logic behind this code. In the first line, we create a variable called sum and then give a value of 0 to this variable. In the second line we define a range in which a counter k will vary. Here k varies from 1 to 100 (Please note the value 101 in the range statement. The starting value is included but the ending value is not included). Also note the colon mark at the end of the statement. It means that the statements following this line with a shift towards right are part of this loop. The last statement is for displaying the value of sum in the screen.

Now let us investigate the series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{100^2}$$

The code will look as below

```
sum=0
for k in range(1,101):
    sum = sum + 1.0/(k**2)
sum
```

What is the series is like the following

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots - \frac{1}{100^2}$$

The code will have to include alternate plus and minus. This can be easily done by considering the general equation of the n^{th} term which is

$$\frac{(-1)^{n+1}}{n^2}$$

```
sum=0
for k in range(1,101):
    sum = sum + (-1)**(k+1)/(k**2)
sum
```

Check the sum of the above series with larger number of terms. Investigate what happens.

Activity B.3 Playing with factors

Now we will consider another example for finding the proper factors of an integer. Proper factors are the factors including the number and 1. For example the number 10 will have four proper factors 1,2,5,10.

The Python program to find the factors of the number n will look like the following.

```
factors=[]
n=10
for k in range(1,n+1):
    if n%k == 0:
        factors.append(k)
```

Initially we declare the variable factor as an empty list (Please note the opening and closing of square brackets). The number whose factors are to be found out is stored in n. In this case we find the factors of 10. Then the counter k is changed from 1 to n and the remainder is found when n is divided with k (The % operator finds the remainder). If the remainder is zero then it means that k is a factor of n and it is added to the list of factors (The append statement does this). After running the program if we type factors and press the enter key, we would get the list of factors. If we want to find the number of factors, then the length of this list will do it which can be found as follows.

```
len(factors)
```

If we are interested in listing out all the factors of all numbers from 1 to 100, then the above program is to be modified by putting an extra loop in the outside to change the numbers from 1 to 100. The resulting code will be as follows

```
for m in range(1,101):
    factors=[]
    num=0
    for k in range(1,m+1):
        if m%k == 0:
            factors.append(k)
    num = len(factors)
    print(m, factors ,num)
```

This program will print the number m, its factors and the number of factors for m when m varies from 1 to 100.

Another interesting exploration possibility is the factors of special sequences. If we need to come up with the factors of the triangular numbers, for instance, the above program can be twisted to suit our needs. Instead of finding the factors of integers from 1 to 100 we need to find the factors of the first 100 triangular numbers. As soon as the first loop is entered, we need to compute the triangular number which is nothing but the sum of all consecutive integers till a particular number.

$$T_n = 1 + 2 + 3 + \cdots + n$$

The resulting program is shown below

```
numfactortriang=[]
    for m in range(1,101):
        triang=0
        for n in range(1,m+1):
                triang=triang+n
        factors=[]
        num=0
        for k in range(1,triang+1):
                if triang%k==0:
                        factors.append(k)
        num = len(factors)
        numfactortriang.append(num)
        print(m,triang ,factors ,num)
    print("Largest no of factors",max(numfactortriang))
```

The variable triang has the triangular numbers which is computed using the peice of code

```
triang=0
for n in range(1,m+1):
        triang=triang+n
```

numfactortriang is a variable which holds the number of factors of the first 100 triangular numbers. So before the start of the program it is initialised to a null variable which holds nothing. As and when the factors are computed and the number of factors calculated, the number is appended to the variable.

In the end of the program, the maximum value of this list is computed and it can be seen that it is 40. So the program gives a very interesting answer to the question what is the largest number of factors for a triangular number below 100. From the list of triangular numbers it can be seen that there are 40 factors for the 95^{th} traingular number 4560.

Activity B.4 The Collatz sequence

Let us look at another example. Here we repeatedly do a process on a number. We start with a number. If the number is an odd number we multiply with three and add one. If the number is an even number, we divide by 2. Then we do this on the result and so on. Let us check what happens to numbers geneated for a starting number, say 13. If we reach the number 1 during any stage of the process, then since it is an odd number multiplying 1 by 3 and adding 1 will give four, which will be divided by 2 and again 2 in the next steps to give 1. So we can safely say that the process is complete when 1 is reached. The code looks like this:

```
a=13
while a!=1:
    if a%2==0:
        a=a//2
    else:
        a=3*a+1
    print(a)
```

Here we use a loop which is called as the while loop. The loop continues its execution until the condition next to the while loop is true. When the condition becomes false, the loop ends.

We can see that the number eventually settles down to 1. Start with a different number and see what happens. Although it is not yet proved, it is thought that all starting numbers will eventually end in 1. This sequence is called the Collatz sequence.

Suppose we need to count the number of steps taken to reach 1, then what can be done to the code to make it capable of this?

A variable to count the number of steps is to be employed as shown below.

```
a=13
count=1
while a!=1:
    if a%2==0:
        a=a//2
    else:
        a=3*a+1
    count=count+1
print(count)
```

Suppose we need to find the steps taken to reach 1 for all numbers from 2 to 100, then a loop for varying a number from 2 to 100 is to be put in the beginning. The code is shown below.

```
for k in range(2,101):
    a=k
    count=1
    while a!=1:
        if a%2==0:
            a=a//2
        else:
            a=3*a+1
        count=count+1
    print(k,count)
```

It can be seen that there are some numbers which take pretty long to reach one. The largest chain length for a starting number below 100 is observed for the number 97 who takes 119 steps to reach 1.

Activity B.5 A taste of primes

Yet another exploration possibility is prime numbers. Mathematicians have been playing around with prime numbers for centuries and the game is still on. The only difference is that the tools used for the game in ancient times were a paper, a pencil and a brain whereas in the modern times it is the computer and a brain.

How do we check whether a number is prime or not? Divide it by all the numbers above 1 and below the number. If any of the numbers divide perfectly, then the number is not prime. Let us try it out through Python.

```
num=13
primeflag=0
for k in range(2,num-1):
    if num%k==0:
        primeflag=1
        break
if primeflag==1:
    print("Not Prime")
else:
    print("Prime")
```

This program checks whether the number 13 is prime or not. If you need to check some other number change the value of num to that number and run the program. In the program, primeflag is a variable that has an initial value of 0. If the program comes across a number that is a factor, then its value is made 1. The statement if num%k==0 checks whether num is exactly divisible by k. Finally if the value of primeflag is 1 then "not prime" is printed in the screen. Otherwise "prime" is printed.

A natural academic question that pops up here is do I need to divide until I reach the number? Can I stop my check below some other number? If we look at the factors of all the composite numbers, no factor is greater than half of the number. This idea can be used here effectively. So instead of dividing until the number, we need to check until half the number. The program can be changed as below

```
num=13
primeflag=0
for k in range(2,num//2):
    if num%k==0:
        primeflag=1
        break
if primeflag==1:
    print("Not Prime")
else:
    print("Prime")
```

Improving the number of divisions are important especially when the number whose primality has to be checked is really large. In areas like cryptography and network security, these numbers are really huge where the number of steps to arrive at the question related to primality is crucial.

In this line, can we improve our program further by reducing the number of checks? Or in other words, can we do better? Again a look at the factors of composite numbers, we can understand that the biggest factor is always less than the square root of the number. This information can be incorporated into our program to get

```
from math import *
num=13
```

```
primeflag=0
for k in range(2,int(sqrt(num))):
    if num%k==0:
        primeflag=1
        break
if primeflag==1:
    print("Not Prime")
else:
    print("Prime")
```

In the above program since the square root is a decimal number we need to convert it to an integer value which is achieved by the int function.

Now suppose we want to print all the prime numbers below 100. What change should be made to the above program? Instead of using the number 13, we need to use the numbers from 2 to 100 which can be done using a for loop. And instead of printing that the number is prime or not it is enough to print the prime numbers alone. So the program can be changed as follows:

```
for num in range(2,101):
    primeflag=0
    for k in range(2,int(sqrt(num))):
        if num%k==0:
            primeflag=1
            break
    if primeflag!=1:
        print(num)
```

There are a number of questions that can be attempted through programs. Some sample challenging questions are given below which can be attempted through programming. Try it out yourself or through your students. The results are really refreshing.

1. If we look at the prime number 41 it can be expressed as a sum of consecutive primes as follows
$$41 = 2 + 3 + 5 + 7 + 11 + 13$$
This number is the longest sum of consecutive primes which is also a prime below 100. Can you write a program to find the losgest sum of consecutive primes below 1000.

2. The sum of primes below 10 is
$$S_{10} = 2 + 3 + 5 + 7 = 17$$
Write a program to display the sum of all primes below 10,100,1000,10000 and so on to check if there is any pattern with these sums

3. The number 197 is a circular prime because all rotations of 197 i.e, 971,719 are all prime. There are thirteen such numbers below 100. Write a program to display them. Extend the program to display all circular primes below 1000.

4. The number 3797 has a very interesting property. It is a prime number by itself. If digits are removed from left to right one at a time, then the resulting numbers 797,97 and 7 are prime. If we do it from right to left, 379,37 and 3 are prime. Write a program to list out all numbers below 10000 which has this remarkable property.

Activity B.6 Square of Digits

As a last exploration exercise in the world of numbers let us attempt the following. A number chain is created by continuously adding the square of digits of the number to form a new number. For example if we start with the number 44, we get the following chain

$$44 \rightarrow 32 \rightarrow 13 \rightarrow 10 \rightarrow 1 \rightarrow 1$$

You can see that the number 1 repeats again and again.

Let us try this with a different number. Say 85

$$85 \to 89 \to 145 \to 42 \to 20 \to 4 \to 16 \to 37 \to 58 \to 89 \to$$

We can see that the number 89 has come up again and the cycle will repeat itself.

What is remarkable with this process is that whatever number we start with, it will either end in 1 or end in 89. Let us write a program to display all such chains with a particular starting number.

```
num=44
checkval=0
while(checkval==0):
        a=[int(x) for x in str(num)]
        sumnum=0
        for y in a:
                sumnum=sumnum+y**2
        if sumnum==1 or sumnum==89:
              checkval=1
        else:
              num=sumnum
              print(sumnum)
```

After the while statement, we are splitting the number into a list of its digits using the statement

```
a=[int(x) for x in str(num)]
```

The idea here is to consider the number 44 as a string (which is considered like a word here) and then going through each letter in the word to get the individual digits. Then if the sum is 1 or 89, we stop the program by putting the value of checkval as 1. (This can be achieved through the break statement also).

Change the starting number from 44 to 89 and see what happens. Check this for anu other number and see the result for yourself.

As a challenging exercise in this line, can you write a program to list out all the numbers below 100 which will reach 89 and not 1?

Use of Python in number theory is a very enriching job. But it is not limited to the number business alone. In the following section, we through light to the use of Python in other areas of mathematics as well.

Activity B.7 Other Areas of Mathematics

Python can also be used in other areas of mathematics, from which some glimpses will be attempted next to show the possibilities.

We will attempt to find limits using Python. For this we need the module sympy which has the defenitions for functions that are used in symbolic mathematics. In some systems it may not be installed so you will have to install it using the following command from the terminal.

```
sudo apt-get install python3-sympy
```

Once it is installed, to find the limit of the function $\frac{1}{x}$ when $x \to \infty$ we use the following code

```
from sympy import *
x = Symbol('x')
l = Limit(1/x,x,S.Infinity)
l.doit()
```

Let us see the meaning of this code. The first line says to import all (* stands for all) functions that are defined in the module sympy for our use. The second line defines that the variable x will now be used as asymbol for a function. The Limit function now evaluates the limit of 1/x where x is actually a symbol and it approaches infinity. Note the last argument S.Infinity which means that the defenitions for Infinity is present in the set S which is defined in the module sympy. The last line doit() is a method which actually computes the value for the limit.

We can try out other functions as well. For example to compute the limit of $\frac{\sin(x)}{x}$, the following command can be used

```
l = Limit(sin(x)/x,x,0)
l.doit()
```

Not only limits, we can do derivatives as well. To compute the derivative of a function the following code can be used

```
d = Derivative(5*x**2+2*x+8,x)
d.doit()
```

We get the derivative as $10x + 2$. This derivative can be found out in a single step instead of two steps. The following code will give the derivative in a single step.

```
Derivative(5*x**2+2*x+8,x).doit()
```

Similarly integrals can be found out as follows

```
Integral(10*x+2,x).doit()
```

will give $5x^2 + 2x$ To find the defenite integral

$$\int_0^2 10x + 2dx$$

we need to enter the command as follows

```
Integral(10*x+2,(x,0,2)).doit()
```

Instead of merely using x we will be using the symbol x as the first argument, the lower limit as the second argument and the upper limit as the third argument.

Python is a very powerful tool in the teaching learning process. Teaching Pyhton in a single small chapter is an impossible task. The aim of this chapter was not to teach Python, but to show that it is learnable and also useful in mathematics instruction.

Appendix C

An insight into the Observation Book

IT Maths lab altogether is a new concept which has no precedence or reference. Nothing different with the Observation Book either. So for writing the Observation Book we need not follow the way in which the other subjects are dealing with their record books. The Observation Book should be an abstract of what the student had gone through in a particular lab work. The learner should be given freedom in writing the OB. But it should include some essential details like aim, observations, details of questions which are marked by note book symbol in lab manual, completed tables, conclusion and procedure.

- Depending upon the nature of the concepts discussed in each lab, the structure of the observation book may vary.
- Space should be provided to write the name of lab and date.
- There should be an aim for each lab. This can be copied from the manual.
- Questions marked with note book symbol and its detailed answer.
- Completed tables.
- Calculations used to find the answer. For example, there will be questions like verify the findings using GeoGebra. Here the details of such calculations should be mentioned.
- Sometimes there will be extended discussions of concepts beyond the lab manual. These should be recorded in the observation book.
- The procedure of a lab activity should be written in the end. It should be an abstract of the steps of each activity and not a copy of the lab manual procedure. It should be written in such a way that a revisit to the OB gives the learner a clear idea of the lab. This may be as follows :

LAB 1

Aim

- To construct an applet to establish geometrically the correspondence of a number and its image under a function.
- To use this applet to find the images of numbers under various functions
- To use an applet to visualise the comparison of a function with an input-output machine.

Activity 1.1

Observations

- When A moves along x axis, C moves along y axis.
- $f(a)$ is the image of the real number a under the function $f(x)$.

- The y coordinate of C is the square of x coordinate of A, since here the function is $f(x) = x^2$.
- The point B lies on the graph of the function $f(x) = x^2$. Then writes the values of $2.3^2, -1.8^2, 0.9^2, 2.9^2$

Activity 1.2

Observations

Here student identifies the functions and then estimate the corresponding values.
The completed tabular column is recorded here.
Comments to the activities asked with the OB icon are also recorded here.
This may be as follows ...

- For the function $f(x) = \frac{1}{x}$ When a approaches to 0 from left side, $f(a)$ decreases to $-\infty$. When a approaches to 0 from right side, $f(a)$ increases to ∞ .
- For the function $f(x) = [x]$, When A moves between two integers, C stays on the least integers among them. When A moves to next interval C jumps to next integer.

Activity 1.3

Observations

Here students finds the values using the given applet ML1.1 and are recorded here . Their comments are also recorded . This may be as follows.

- For the function $f(x) = sqrt(x)$, if we give a negative number as the input, the warning light of the machine turns red. This means that negative numbers are not in the domain of the function $f(x) = sqrt(x)$.
- Domain of the function $f(x) = sqrt(x)$ is $[0, \infty)$
- In a similar manner they can write the case of the function $f(x) = \frac{1}{x}$

Conclusion

Here students consolidate the findings in the above three activities.

- The set of points $(x, f(x))$ represents the graph of the function $f(x)$
- Graph of a function may be used to find the value $f(x)$ for a given x
- We can consider a function as an input - output machine.
- If we give a number x, which is in the domain of f, we get $f(x)$ as the output.
- If x is not in the domain of f, output is not produced. Which means that $f(x)$ is not defined.

Procedure

Here students write a brief description of what they have done in the lab.
An example is given below.

- An applet is constructed using the concept that the set of points (x,f(x)) represents the graph of the function f(x).
- This applet is used to find values of some functions at given points.
- Behaviour of some functions are also discussed with the help of the applet.
- Using 'Function Machine', values of some functions at some points are evaluated.

Appendix D

Model Practical Evaluation Questions

Here a few model questions are provided in connection with the practical evaluation of the lab activities. The questions of scores 2, 4 and 6 are provided. For the practical examination a question of score 8 (a suitable combination of 2 , 4 and 6) will be asked from each lab activity . The evaluation of Practical Examination focuses only on the mathematical concepts rather than the technical side of GeoGebra. These model questions will definitely provide a better awareness to the teacher how a question can be framed from a particular lab activity, which in turn helps the teacher to carry out the lab activities in the right direction.

2 Mark Questions

1. Plot the graph of $f(x) = -|x| + 1$ (input: `-abs(x)+1`) (LAB-1)
   (a) Find the maximum value of f(x)
   (b) Find the minimum value of $f(x)$ in $x \in [-2, 1]$.
2. Using the given applet Q2.4 answer the following questions. (LAB-2)
   (a) Identify the point at which the given function $f(x)$ is not defined.
   (b) Shift the above graph using the sliders in such a way that it is not defined at $x = 2$ and range becomes $\mathbb{R} - \{2\}$. Write the new function in terms of f.
3. Using the input command `f=If[x<=1,x,3]` draw the graph of the function f. (LAB-3)
   Write the function f and find its domain and range
4. Use applet Q4.2 (LAB-4)
   Using the slider you can change the rotation of the point P along the unit circle. Arrange the following values in increasing order .

   $\sin 0$, $\sin 1$, $\sin 2$, $\sin 3$, $\sin 4$, $\sin 5$
5. Use applet Q5.1 (LAB-5)

   Find the values of the following with the help of the applet

   You can change the rotation of the point A using the input box. Coordinates of A are not visible. From the coordinates of P, estimate the required values (A and P are diametrically opposite points)

   i) $\sin 2.7$ ii) $\cos 3.7$
6. Plot the graph of the function $f(x) = \sin 2x$. Find the solution of the equation of $\sin 2x = 0$ in the interval $(0, 2\pi)$. (LAB-6)

7. Using the applet Q7.5,complete the following table (LAB-7)

Function	Range	Length cycles	Period
$\cos x$	$[-1,1]$	2π	2π
$\cos\frac{x}{2}$			
$\cos\frac{x}{3}$			

8. Using a given applet ,write the normal form of the line $\sqrt{3}x - y + 8 = 0$ (LAB-8)

9. Use the Applet (LAB-9)

   In the given applet when we play the animation play button the points C and D move. Observe the changes of the distance AD and BD when D and C move .Stop the animation at any three different places and fill the table.

Position	Distance AC or AD	Distance BC or BD	Sum AD+BD	Sum AC+BC
Position 1				
Position 2				
Position 3				

   Name the path formed by movements of C and D together.

10. By giving proper input Draw the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$. (LAB-11)
    Find its foci and check it with the command to find the foci of hyperbola .

11. Draw a line segment AB which is internally divided by the xy plane in the ratio 2:3. Write its coordinates. (LAB-12)

12. Use Applet Q16.3 (LAB-16)

    You can change the radius of the circle and the angle using the sliders or input boxes. Find the position of the following complex numbers And complete the table.

    i) $\sqrt{3}+i$ ii) $-1-\sqrt{3}i$

Complex Number	Amplitude	Argument
$\sqrt{3}+i$		
$-1-\sqrt{3}i$		

13. Find the centre and radius of the circle $x^2+y^2-4x-8y-45=0$. Draw the circle and verify your answer. (LAB-10)

4 Mark Questions

14. Using applet Q1.2, complete the following table (LAB-1)

Question	Function $f(x)$	Value of x	Value of $f(x)$
$\sqrt{1.8}$			
$(3.46)^{\frac{-3}{2}}$			
$\sqrt{\sqrt{5}}$			

15. Use the given applet Q2.2 to answer the following (LAB-2)
    Graph of the function $f(x) = |x|$ is given in Graphics 2

(a) Get the graph of $g(x) = -|x+2| + 1$ from the graph of f, by suitable reflections and translations. You can use the buttons given in the Graphics view. (2)

(b) Write the domain and range of $g(x)$. (1)

(c) Find the maximum value of $g(x)$. (1)

16. Plot the graph of the function given below (LAB-3)

$$f(x) = \begin{cases} 1, & \text{if } \quad x < -1 \\ x^2 & \text{if } \quad -1 \le x \le 1 \\ -1 & \text{if } \quad x > 1 \end{cases}$$

(Input :`If[x<-1,1,-1<=x<=1,x^2,-1]`)

(a) Find the value of $f(\frac{1}{2})$ (1)

(b) Find the range of f (2)

(c) Plot the graph of $g(x) = f(x) + 1$ and find the range of g (1)

17. Draw the graphs of the following functions and complete the table (LAB-4)

(a) $\sin x$ (b) $\tan x$ (c) $\sec x$

Function \ Interval	$(0, \frac{\pi}{2})$	$(\frac{\pi}{2}, \pi)$	$(\pi, \frac{3\pi}{2})$	$(\frac{3\pi}{2}, 2\pi)$
$\sin x$		decreases from 1 to 0		
$\tan x$				increases from $-\infty$ to 0
$\sec x$	increases from 0 to ∞			

18. Use applet Q5.3 (LAB-5)

Complete the following table with the help of the applet

Note : You can change the rotation of the point A using the input box. Rotation of the point P can be selected by clicking on the text boxes. Coordinates of A are not visible and that of P are visible only if it is in the first quadrant. Select a suitable rotation of P so that you can estimate the required values using its coordinates.

	$\cos 2.5$	$\sin 1$	$\cos 5.5$	$\sin 4$
Rotation of P				
Value				

19. Use Appllet 6.3 (LAB-6)

Complete the following table.
(use the slider n to change the x value marks in x axis. Input f and g functions in input box . Input the corresponding function to get the graph)

Equations	Points of intersection in $[0, 2\pi)$	Principal value solutions	General Solutions
$\cos x = \sin x$			
$\sin 2x = \cos 3x$			

20. Using the applet Q7.3,complete the following table.(LAB-7)

Function $a * f(x)$	Range	No:cycles in $[0, 2\pi]$	Period
$\sin x$	$[-1, 1]$	1	2π
$2 \sin x$			
$0.5 \sin x$			
$\sin 2x$			
$\sin 3x$			

21. Use the Applet 8M.1 (LAB-8)

    Equation of the given line is $ax + by + c = 0$. You can change the values of a, b and c using input boxes.

    Draw the lines, satisfying the following conditions, using the applet and complete the given table.

    i) Passing through the origin, slope is $\frac{3}{2}$

    ii) x intercept 3 and y intercept -4

    iii) Passing through the points $(1, 1)$ and $(3, 5)$

    iv) y intercept 3 and slope -2

Line	a	b	c
i			
ii			
iii			
iv			

22. Read the construction given below carefully and answer the following question. (LAB-9)

    - Create the slider a with Min=0
    - Plot the point $A(4, 0)$
    - Draw the line $x = a$ and draw the circle of radius a centered at A
    - Mark the points of intersections C and D of the circle with the line.
    - Animate the slider

    (a) Write the reason that the curve traced is a Parabola. Write its foci and directrix (2)

    (b) Construct the applet and verify your answer (2)

23. Find the focus and length of latus rectum of the parabola $x^2 = -8y$ and verify the answer geometrically (LAB-10)

24. Draw the following ellipse using Ellipse tool (LAB-11)

    (a) Foci $(\pm 4, 0)$,passing through $(5, 2)$

    (b) Foci $(\pm 3, 0)$ at length of major axis is 10.Also find the length of LR of the ellipse geometrically

25. Construct a cube such that one of its face lies in the first octant and the opposite face is in the second octant. Write its co-ordinates. (LAB-12)

26. Using the given applet Lab 13.1 (LAB-13)

    (a) Plot the graph of the following piece wise function and check the limit at x=2.

    $$f(x) = \begin{cases} x^2 & \text{if } x \leq 2 \\ 2x + 1 & \text{if } x > 2 \end{cases}$$

    (Input `If[x<=2,x^2,2x+1]`)

(b) How to change the above function such as a limiting function at x=2 .Plot the same.

27. Use the Applet Q14.2 (LAB-14)

Graph of a function f and a point P on it is given. You can change the position of the point using the slider x or the input box. Tangent to the curve at P and its slope is also given. Answer the following Questions.

(a) Find the derivative of the function at $x = 0$ and $x = 2$. (2)

(b) Find the points on the curve at which the tangent is parallel to the x axis. Hence find the values of x at which $f'(x) = 0$. (2)

28. Draw the solution region of the system of linear inequalities (LAB-16)

$$\begin{aligned} x - 2y &\leq 3 \\ 3x + y &\geq 12 \\ x &\geq 0 \\ y &\geq 1 \end{aligned}$$

6 Mark Questions

29. Consider the parabola $y^2 = 12x$ (LAB-10)

(a) Find the focus and equation of directrix, then draw the parabola (2)

(b) Find the co-ordinates of the end points of latus rectum (2)

(c) Draw the circle passing through the vertex and end points of the latus rectum without using *circle through 3 points* tool (2)

30. Using applet Q1.1, answer the following questions. (LAB-1)
Move the point A using the slider **a** and observe the graph

(a) Write the function $f(x)$ representing the graph. Plot the function by giving suitable input of the function . (2)

(b) Using the graph of $f(x)$ estimate the value of x ,if (2)
(a) $f(x) = 12$
(b) $f(x) = 21$

(c) Find the intervals in which (2)
(a) $f(x) > 0$
(b) $f(x) < 0$

31. Consider the graph of the function $f(x) = x^2$ (LAB-2)

(a) Write the function $g(x)$ obtained by shifting the graph of $f(x)$ by 4 units vertically downwards. Draw the graph of g and find its points of intersection with x axis. (2)

(b) Find the function function h, obtained by shifting the graph of f, such that the graph of h passes through the points $(-\sqrt{2}, 0)$ and $(\sqrt{2}, 0)$. Draw its graph. (1)

(c) Plot the graph of $-h$ and find the maximum value of $-h(x)$ (1)

(d) If the graph of f is shifted 2 units towards right (Horizontally) and then 3 units upwards (Vertically) ,write the function of the transformed graph. (2)

32. Use the Applet Q3.4. Graph of a function f is given in the applet. Some more functions are also defined (g, h, ...). You can show/hide their graphs from algebra window. (LAB-3)

(a) Identify the functions among g, h, ... which are equal to f in some intervals. (2)

(b) Write the definition of f and plot its graph. (2)

(c) Find the domain and range of f. (2)

33. Use the given applet Q4.1. (LAB-4)

    You can change the rotation of the point P by changing x using the input box.

    (a) Complete the following table. (4)

Function \ x	1.2	2.1	10	-0.5
$\sin x$				
$\cos x$				

    (b) Find the values of $\sin(\frac{21\pi}{5})$ and $\cos(-\frac{30\pi}{7})$. *Note that the input box accepts the value of x between -10 and 10 only. You are not allowed to edit the properties of the slider* (2)

34. Use applet Q5.5 (LAB-5)

    Complete the following table with the help of the applet

    Note : You can change the rotations of the point A and P using the input boxes. Coordinates of A are not visible and that of P are visible only if it is in the first quadrant.
    Give a suitable rotation for P so that you can estimate the required values using its coordinates

	$\sin 2.7$	$\cos 2.7$	$\sin 5.3$	$\cos 4.2$	$\sin 1.3$	$\cos 1.3$
Rotation of P						
Value						

35. Using the applet Lab 6.1, complete the following table. (LAB-6)

    (use the slider n to change the x value marks in x axis)

Equations	Points of intersection in $[0, 2\pi)$	Principal solutions	General solutions
$\sin x = \frac{1}{2}$			
$\sin 3x = -\frac{\sqrt{3}}{2}$			
$\sin \frac{x}{3} = \frac{\sqrt{3}}{2}$			

36. Each of the function given in List 1 matches with one of the function given in List 2. Draw the graph of the functions given in List 1, observe the graph and identify the function from List 2 which represents the same graph. You can use the Applet Q7.1 (LAB-7)

    List-1: $\sin(x-2\pi)$, $\cos(x-\frac{3\pi}{2})$, $\cos(x+\frac{\pi}{2})$, $\sin 2(\frac{\pi}{4}-x)$, $\cos 4(\frac{3\pi}{8}+x)$, $\sin \frac{1}{2}(3\pi - x)$

    List-2: $\sin x$, $\cos x$, $-\sin x$, $-\cos x$, $\cos\frac{x}{2}$, $-\cos\frac{x}{2}$, $\sin 2x$, $\cos 2x$, $\sin 4x$, $\cos 4x$

37. Applet Q8.1 is given (LAB-8)
    A line of the form $ax + by + c = 0$ is given in the applet .You can change the value of a,b and c using input box .

    (a) Match the following

A	B
i) a=0 b=4 c=3	1)x intercept 3,y intercept 4
ii) a=3 b=0 c=3	2)passes through origin
iii) a=5 b=2 c=0	3)parallel to X axis
iv) a=4 b=3 c=12	4)parallel to Y axis

(b) Give proper values for a,b and c so that slope of the line becomes
(i)$\frac{1}{2}$ (ii)-3
(Check it with slope tool)

38. Use Applet (LAB-9)

Using the slider you can change the angle β made by the plane with the axis of the cone.

(a) Find the values of β for which the following curves are obtained as the intersection of the plane with the cone (4)

Curve	β
Circle	
Parabola	
Ellipse	
Hyperbola	

(b) Find the semi vertical angle of the cone. (1)

(c) Find the value of β for which the curve of intersection becomes a pair of intersecting straight lines (1)

39. Using the Hyperbola Tool draw the following hyperbola (LAB-11)

(a) Foci $(\pm 2, 0)$,passing through $(2, 3)$. (1)

(b) Foci $(0, \pm 3)$,length of transverse axis is 4. (2)

(c) Foci $(\pm 5, 0)$ length of conjugate axis 6 . (3)

40. Give the Applet Named Octant 3.ggb in the manual activity 12.4 (LAB-12)

(a) By adjusting sliders ,make co-ordinates of A as $(3, 2, -2)$ and that of B as $(3, 2, -2)$

(b) Name the co-ordinate plane divides the line segment AB internally. What is that ratio?

(c) Adjust the co-ordinates of B by adjusting the sliders so that the line segment AB is divided by the YZ plane internally

(d) Keeping the co-ordinates of A as $(3, 2, 4)$, adjust the co-ordinates of B,so that origin becomes the mid point of AB

41. Use Applet Q13.1 (LAB-13)

Three functions are given. You can select one by one using the buttons provided. Find their limit at a, if it exists, otherwise write the reason for non existence. Write the left and right limits in each case. (If you want you can change the increment of the slider h using the input box).

Function	a	Left limit	Right limit	Limit
Function 1				
Function 2				
Function 3				

42. Use Applet Q14.1 (LAB-14)

Three functions are given. You can select them one by one using the buttons. Find the derivative of each function at a (given) if it exists. If it does not exist, write the reason. In each case write the left derivative and the right derivative.

Function	a	Left derivative	Right derivative	Derivative at a
Function 1				
Function 2				
Function 3				

43. Using the applet Q15.1, answer the following (LAB-15)

(a) $f(x) = x^2 - 4x$ and the tangent to any point on the curve.(By moving the slider, you can move the point as well as the tangent.)

Fill up the following table by moving the slider (4)

<table>
<tr><th>$f(x)$</th><th>Value of a</th><th>Slope of the tangent at a</th><th>$f'(a)$</th><th>$f'(x)$</th></tr>
<tr><td rowspan="6">$f(x) = x^2 - 4x$</td><td>4</td><td></td><td></td><td rowspan="6"></td></tr>
<tr><td>2.5</td><td></td><td></td></tr>
<tr><td>2</td><td></td><td></td></tr>
<tr><td>1.2</td><td></td><td></td></tr>
<tr><td>0</td><td></td><td></td></tr>
<tr><td>-0.5</td><td></td><td></td></tr>
</table>

Using the table, mark the points $(a, f'(a))$ in the geogebra window. Draw the straight line joining these points. Write the function represented by the above graph.

(b) Here, the value of slider a,we used is between -5 and 5.Fill up the following table. (2)

Value of a is	Sign of $f'(a)$
in $[-5, 2]$	
in $[2, 5]$	
at 2	

Appendix E

Geogebra Basic Tools

E.1 Arrangement of Tools

In GeoGebra construction tools are arranged in 12 sets as shown in Figure E.1. All the tools in each set is obtained by clicking on the small arrow at the bottom right corner of each icon as shown in Figure E.2. Keeping the cursor on the tool, a brief description of the function of the tool is displayed.

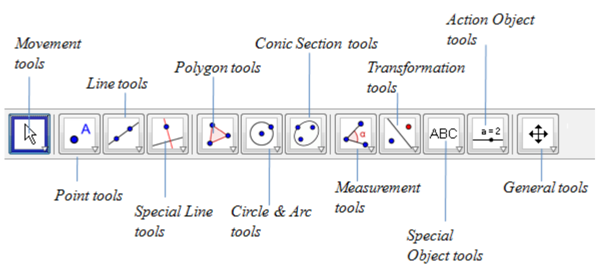


Figure E.1: Sets of Tools

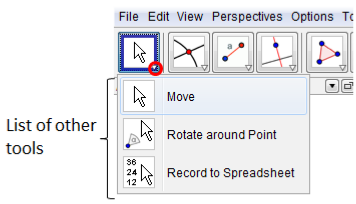
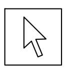
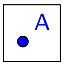
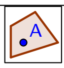
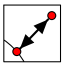
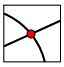
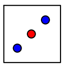
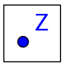


Figure E.2: Sets of Tools

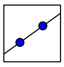
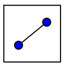
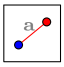
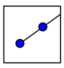
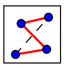
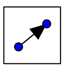
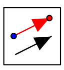
E.2 Movement Tools

	This tool is used for moving Points, Geometrical figures, Graphs etc.

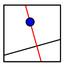
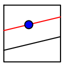
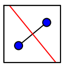
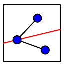
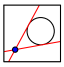
E.3 Point Tools

	Create a new point
	Point on Object - You can create points on other geometrical objects. The point can be moved on the object, but cannot be taken out of the object.
	We can attach a point to an object by clicking on the point and the object. To detach a point from an object, click on the point with the tool.
	This tool is used to mark the point of intersection of curves or Geometrical figures.
	This tool helps in locating midpoint of a line segment or any two points.
	This tool helps in representing a given point in a complex number form.

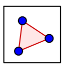
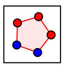
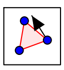
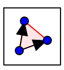
E.4 Line Tools

	A line through two points
	A segment between two points
	Click on the Graphics view after selecting the tool will bring up a popup where length of the segment can be entered.
	To draw a ray starting from the first point and directed through the second point
	Click on multiple points, the last point being on the initial point results in a figure called polyline.
	First click on the Graphics view with `Vector` tool active, would be a starting point of a vector. Second click would be the end point of the vector.
	Create a vector and a point first. Now click on the point and vector in the sequence creates another vector having same magnitude and direction as the earlier vector through the point initially constructed.

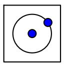
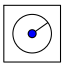
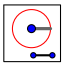
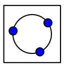
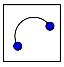
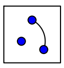
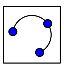
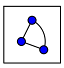
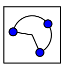
E.5 Special Line tools

	This tool allows us to draw a line perpendicular to a given line and passing through a given point.
	Parallel line to a given line through a point.
	After selecting the tool click on a line segment to get a perpendicular segment to the given line.
	After activating this tool select the three vertices such that the vertex where angle is formed is in the middle. An angle bisector will be drawn.
	Selecting a point A and the graph of a function gives all tangents through A to the function.

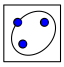
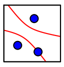
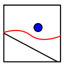
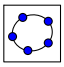
E.6 Polygon Tools

	Successive clicks when the `Polygon` tool is active results in a polygon. The last click should be on the first point to close the polygon.
	The first two clicks would decide the length of the side of a regular polygon. A pop-up window appears. Number of sides in the desired regular polygon need to be entered.
	Identical to the `Polygon` tool except that the polygon formed is rigid.
	This tool generates a polygon with sliders for positioning the vertices.

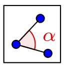
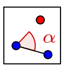
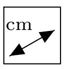
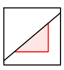
E.7 Circle and Arc Tools

	First click on the Graphics view with `Circle with Center through Point` tool active would be a center of the circle. Second click would be a point on the circle.
	Identical to the first one except that the radius need to be entered numerically in a pop-up window.
	This tool is useful in duplicating a circle. After activating `Compass` tool, select the circle to be duplicated. Click a point where the center of the duplicate circle is expected to be located.
	Three clicks on the Graphics view when the `Circle through Three Points` active would result in a circle through the points identified through the clicks.
	Two clicks on the Graphics view when the `Semicircle through 2 Points` active would result in a semicircle.
	When `Circular Arc` tool is active, three clicks on the Graphics view results in a circular arc. First click defines the center of the arc, second an end point, and third the length of the arc.
	Three clicks define a circumcircular arc with all the three points on the arc.
	When `Circular Sector` tool is active, three click on the Graphics view results in a circular sector. First click defines the center of the sector, second an end point, and third the length of the sector.
	Three clicks define a circumcircular sector with all the three points on the sector.

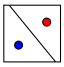
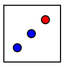
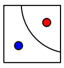
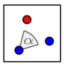
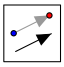
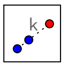
E.8 Conic section Tools

	Three clicks on the Graphics view when `Ellipse` tool is active results in an ellipse. The first two clicks would be the foci of the ellipse. The third click would define the ellipse.
	Three clicks on the Graphics view when `Hyperbola` tool is active results in a hyperbola. The first two clicks would be the foci of the hyperbola. The third click would define the hyperbola.
	Draw a point and a line. With `Parabola` tool active, click on the point and the line makes them focus and directrix resulting in a parabola.
	Five successive clicks on the Graphics view defines a conic through the five points. The location of the fifth point would define the nature of the conic section.

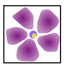
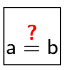
E.9 Measurement Tools

	This tool helps in measuring the angles. Following are the list of angles measurement situations using Angle Tool: Three sequential clicks on the end points of the intersecting line segments with the second click being the vertex where angle is formed. Click on two segments to measure the angle between them. Click on two lines to measure the angle between them. Click on two vectors to measure the angle between them. Click on a polygon to measure all angles of the polygon.
	This tool is useful in creating an angle of desired size. For example, in order to create angle ABC with B = 400, first draw line segment AB, then using Angle with given Size tool, click on A and B successively and enter 400 in the appearing dialogue box.
	This tool gives the distance between two points, distance of a point from a line, length of a line segment, perimeter of polygon, circumference of a circle/ellipse.
	Click on the polygon/circle/ellipse when the `Area` tool activated would result in area display.
	Click on a line/line segment results in display of the slope when `Slope` tool is active.

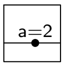
E.10 Transformational Tools

	This tool gives a reflection of an object about a line. Activate `Reflect about Line` then click on the object to be reflected and the line about which the object to be reflected.
	This tool is useful to get the reflection of an object about a point. Click on the object and the point results in the reflection.
	This tool gives the reflection of an object about a circle - click on the object and the circle.
	This tool helps in rotating an object around a point at a desired angle of rotation. Select the object. Click on a point to specify the center of rotation and then enter the rotation angle into the text field of the appearing dialogue window.
	Select the object that we want to translate. Then click on a vector or two points. The object will be translated by the magnitude of the vector in its direction.
	This tool allows us to enlarge/reduce an object by a given factor. Select the object to be dilated. Then, click on a point to specify the dilation center and enter a number (dilation factor) into the text field of the appearing dialogue window. If the number is greater than 1, object will be enlarged. If the number is less than 1 the object will be reduced.

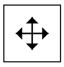
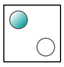
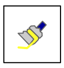
E.11 Special Object Tools

	This tool is useful in creating static and dynamic text in the Graphics View. Activate the `Text` tool. Click on a point to create a new text that is attached to this point. Type the text in the dialogue box.
	This tool is used to insert an image. Click in the Graphics View, when the `Image` tool is active, to specify the position of the image's lower left corner. Then, a file-open dialogue appears that allows us to select the image from the files saved in the computer.
	The Pen Tool allows us to add freehand notes and drawings to the Graphics View. This makes the Pen Tool useful when using GeoGebra for presentations. To erase a portion of the drawing press and hold the right mouse button and move the cursor to the portion of the drawing one need to erase.
	The relation function helps us know a general relationship between two objects. For example, draw two parallel line segments and use Relation tool. The two relation between them as parallel line segments is displayed.

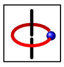
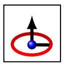
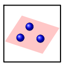
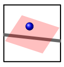
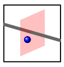
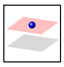
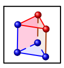
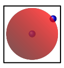
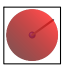
E.12 Action Object Tools

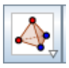
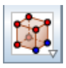
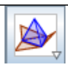
	Slider, can be considered as a variable, which takes any value between two numbers. To create a slider, select the `Slider` tool and click on the Graphics view. There are three options with in it – Number, Angle, and Integer.
	Check box is used to hide/view the objects that are constructed.

E.13 General Tools

	Used to move the Graphics view. Click, hold and drag the drawing pad in the Graphics View to change its visible area.
	Click on any place on the drawing pad to zoom in.
	Click on any place on the drawing pad to zoom out.
	If we click on any object with the tool, it will be hidden from the Graphics view when we select any other tool. To see the object again, select the tool.
	Click on an object to show or hide its label.
	This tool allows us to copy visual properties (e. g., color, size, line style) from one object others. To do so, first select the object whose properties we want to copy. Then, click on all other objects that should adopt these properties.
	Click on any object that we want to delete. Note: We can use the Undo button if we accidentally delete the wrong object.

E.14 Additional Tools for 3D Graphics

	We can construct circles with given axis and passing through a given point - with the tool active, select a line and a point.
	We can construct circles with given centre, radius and perpendicular to a given direction with the tool active select the centre and a vector/two points.
	With the tool active, select two objects (for example two spheres) to produce the curve of intersection.
	With the tool active, select three points to construct the plane passing through them.
	Creates a plane through 3point/through a point and a line/contains two intersecting lines/contains a polygon.
	Creates a plane passing through a given point and perpendicular to a given line or vector.
	Creates a plane which is parallel to a given plane and passing through a given point.
	Creates prism/parallelepiped with given base and top. To construct a prism select a polygon for its bottom and the first point on its top.
	Creates a sphere with given centre and passing through a given point.
	Creates a sphere with given centre and radius.
	Click and drag the 3D graphics view to rotate it.

	Creates Pyramids with given base and vertex by selecting the base polygon and the vertex.
	Using this tool we can create pyramid or cone with given base and height – for this select the base and enter the height.
	Creates a Prism or Cylinder with given base and height
	Creates a cone with given vertex and base radius – choose the centre of the base then the vertex and enter the base radius.
	Choose the centre of top and bottom circles and enter radius to make a cylinder.
	With the tool active, select two points to create a regular tetrahedron with the selected points as two vertices.
	With the tool active, select two points to create a cube with the selected points as two end points of an edge.
	With the help of this tool we can create net of polyhedrons (pyramid, prism etc.). With the tool active, click on a polyhedron to create its net. A slider will appear on the graphics view, with the help of which we can open or fold the net.
	Shows volume of a solid.
	Creates the reflection of an object on a plane.
	With this tool we can rotate an object about a line by an angle – for this click on the object, line and enter angle of rotation.
	To change the view in front of object clicked

Appendix F

Geogebra Commands

F.1 Using Commands

All tools can be used in the command mode also. The commands are to be entered in the input box. While using commands, you need to be very careful while using the brackets. The right form of bracket is very crucial to get the expected result.

F.2 Point Commands

A=(0,0)	Creates a new point A with coordinates $(0, 0)$
Point(<Object>)	Point on Object - If you have a circle whose name is c, then Point(c) will create a point on the circle c. If the newly created point should have a name of your choice, say B, then B=Point(c) will create a point whose name is B and the point B will be on the object c
Intersect(<Object>,<Object>)	Intersect two objects - This command is used to find the intersection of two objects. For example if we have two intersecting circles c and d, then Intersect(c,d) will give the two points of intersection.
Center(<Object>)	Center Command - Using this command creates the center of an object.

F.3 Line Commands

Line(<Point>,<Point>)	Line Command - Line(A,B) Creates a line including the points A and B.
Segment(<Point>,<Point>)	Segment Command - Segment(A,B) Creates a segment that starts from the point A and ends in the point B.
Segment(<Point>,<Length>)	Segment Command - Segment(A,5) creates a segment from the point A with length 5.
Ray(<Point>,<Point>)	Ray Command - Ray(A,B) creates a ray with A as starting point which passes through the point B
Polyline(<Point>,...<Point>)	Create a polyline that includes a number of points
Vector(<Point>,<Point>)	The Vector Command - The command Vector(A,B) creates a vector starting from A and ending in B.
Vector(<Point>)	The Vector Command - The command Vector(A) creates a vector from the origin to the point A.

F.4 Special Line Commands

`PerpendicularLine(<Point>,<Line>)`	Perpendicular Line Command - The command `PerpendicularLine(A,l)` creates a perpendicular line from the point A to the line l. This command will work even if l is a segment.
`Line(<Point>,<ParallelLine>)`	Parallel Line Command - The command `Line(A,l)` will create a line parallel to the line l passing through the point A.
PerpendicularBisector(<Segment>)	Perpendicular Bisector Command - The command `PerpendicularBisector(l)` will creates a line which is a perpendicular bisector to the segment l.
`AngleBisector(<Line>,<Line>)`	The Angle Bisector Command - The command `AngleBisector(m,n)` will create angle bisector to the line m and n. Please note that two bisectors will be formed through this command. If we want only one bisector then the format `AngleBisector(B,A,C)` will create an angle bisector to the angle formed by joining the points A, B and C.
`Tangent(<Point>,<Conic>)`	The Tangent Command - If c is a circel, the command `Tangent(A,c)` creates tangents from the point A to the circel c. c can be a function also.

F.5 Polygon Commands

`Polygon(<Point>,...,<Point>)`	Polygon Command - The command `Polygon(A,B,C,D,A)` creates a polygon (quadilateral) starting from A and ending in A, passing through the points B, C and D. The starting point and the ending point should be the same. For creating an n sided polygon, there should be $n+1$ points.
`Polygon(<Point>,<Point>,<Number of Vertices>)`	Regular Polygon Command - The command `Polygon(A,B,5)` creates a pentagon with the points A and B as the end points of one side.
`RigidPolygon(<Polygon>)`	Rigid Polygon Command - The command `RigidPolygon(p)` will make a rigid copy of the polygon p which is not a rigid polygon.

F.6 Circle and Arc Commands

Circle(<Point>,<Point>)	Circle Command - The entry of Circle(A,B) creates a circle with center A passing through the point B.
Circle(<Point>,<Radius>)	Circle Command - The entry of Circle(A,5) creates a circle with center A having a radius of 5 units.
Circle(<Point>,<Point>,<Point>)	Circle Command - The entry of Circle(A,B,C) creates a circle passing through the points A, B and C.
SemiCircle(<Point>,<Point>)	Semi Circle Command - The entry of Circle(A,B) creates a semi circle through two points A and B where the length AB is the diameter.
CircularArc(<Mid Point>,<Point>,<Point>)	Circular Arc Command - The entry of CircularArc(A,B,C) creates a circular arc whose center is at A and the end points are B and C.
CircularSector(<Mid Point>,<Point>,<Point>)	Circular Arc Command - The entry of CircularSector(A,B,C) creates a circular arc whose center is at A and the end points are B and C.

F.7 Conic section Commands

Ellipse(<Focus>,<Focus>,<SemiMajor>)	Ellipse Command - The entry of Ellipse(A,B,5) will draw an ellipse whose focus points are A and B with a semi major axis of 5
Ellipse(<Focus>,<Focus>,<Segment>)	Ellipse Command - The entry of Ellipse(A,B,m) will draw an ellipse whose focus points are A and B with a semi major axis of length same as the length of the segment m
Ellipse(<Focus>,<Focus>,<Point>)	Ellipse Command - The entry of Ellipse(A,B,C) will draw an ellipse whose focus points are A and B passing through the point C.
Parabola(<Point>,<Line>)	Parabola Command - The entry of Parabola(A,m) will give a parabola whose focus is the point A and the directrix is the line m.
Hyperbola(<Focus>,<Focus>,<SemiMajor>)	Hyperbola Command - The entry of Hyperbola(A,B,5) will draw a hyperbola whose focus points are A and B with a semi major axis of 5
Hyperbola(<Focus>,<Focus>,<Segment>)	Hyperbola Command - The entry of Hyperbola(A,B,m) will draw a hyperbola whose focus points are A and B with a semi major axis of length same as the length of the segment m
Hyperbola(<Focus>,<Focus>,<Point>)	Hyperbola Command - The entry of Hyperbola(A,B,C) will draw a hyperbola whose focus points are A and B passing through the point C.

F.8 Measurement Commands

`Angle(<Vector>)`	Angle Command - The use of `Angle(v)` will give the angle between the vector v and the x axis.
`Angle(<Vector>,<Vector>)`	Angle Command - The use of `Angle(u,v)` will give the angle between the vectors u and v.
`Angle(<Line>,<Line>)`	Angle Command - The use of `Angle(m,n)` will give the angle between the lines m and n.
`Angle(<Line>,<Plane>)`	Angle Command - The use of `Angle(m,p)` will give the angle between the line m and plane p.
`Angle(<Plane>,<Plane>)`	Angle Command - The use of `Angle(p,q)` will give the angle between the planes p and q.
`Angle(<Point>,<Apex>,<Point>)`	Angle Command - The use of `Angle(A,B,C)` will give the value of angle $\angle ABC$
`Distance(<Point>,<Line>)`	Distance Command - The use of `Distance(A,m)` will give the distance between the point A and the line m.
`Distance(<Line>,<Line>)`	Distance Command - The use of `Distance(m,n)` will give the distance between the lines m and n.
`Distance(<Plane>,<Plane>)`	Distance Command - The use of `Distance(p,q)` will give the distance between the planes p and q.
`Area(<Polygon>)`	Area Command - The use of `Area(p)` will give the area of the polygon p.
`Slope(<Line>)`	Slope Command - The use of `Slope(m)` will give the slope of the line m.

F.9 Transformational Commands

`Reflect(<Object>,<Point>)`	Reflect Command - The use of `Reflect(p,A)` reflects the object p through the point A.
`Reflect(<Object>,<Line>)`	Reflect Command - The use of `Reflect(p,m)` reflects the object p about the line m.
`Reflect(<Object>,<Circle>)`	Reflect Command - The use of `Reflect(p,c)` inverts the object p with respect to the circle c.
`Reflect(<Object>,<Plane>)`	Reflect Command - The use of `Reflect(p,q)` reflects the object p about the plane q.
`Rotate(<Object>,<Angle>)`	Rotate Command - The use of `Rotate(p,a)` will rotate the object p by an angle a.
`Rotate(<Object>,<Angle>,<Point>)`	Rotate Command - The use of `Rotate(p,a,A)` will rotate the object p by an angle a around the point A.
`Rotate(<Object>,<Angle>,<Axis>)`	Rotate Command - The use of `Rotate(p,a,m)` will rotate the object p by an angle a around the axis of rotation m.
`Translate(<Object>,<Vector>)`	Translate Command - The use of `Translate(p,v)` translates the object, say polygon p by the vector v.
`Translate(<Vector>,<Point>)`	Translate Command - The use of `Translate(v,A)` translates the vector v to the start point A.
`Dilate(<Object>,<Factor>)`	Dilate Command - The use of `Dilate(p,2)` enlarges the object p by a factor 2. Use of a number less than 1 will shrink the object.

F.10 3D Commands

Pyramid(<Polygon>,<Point>)	Pyramid Command - The use of Pyramid(p,A) creates the pyramid by using the polygon p as the base and the point A as the vertex.
Pyramid(<Polygon>,<Height>)	Pyramid Command - The use of Pyramid(p,3) creates the pyramid by using the polygon p as the base and height 3.
Prism(<Polygon>,<Point>)	Prism Command - The use of Prism(p,A) creates the prism by using the polygon p as the base and the point A as the vertex.
Prism(<Polygon>,<Height>)	Prism Command - The use of Prism(p,3) creates the prism by using the polygon p as the base and height 3.
Cone(<Circle>,<Height>)	Cone Command - The use of Cone(c,3) will create a cone with base as the circle c and height 3.
Cone(<Center>,<Vertex>,<Radius>)	Cone Command - The use of Cone(A,B,5) will create a cone with base as the circle with center A and radius 5 and the vertex is the point B.
Cylinder(<Circle>,<Height>)	Cylinder Command - The use of Cylinder(c,3) will create a cylinder with base as the circle c and height 3.
Cylinder(<Center>,<Vertex>,<Radius>)	Cylinder Command - The use of Cylinder(A,B,5) will create a cylinder with bottom center A and top center B with radius 5.
Cube(<Square>)	Cube Command - The use of Cube(s) will create a cube with the square s.
Cube(<Point>,<Point>,<Point>)	Cube Command - The use of Cube(A,B,C) will create a cube with vertices A, B and C.
Sphere(<Center>,<Radius>)	Sphere Command - The use of Sphere(A,5) will create a sphere with center A and radius 5.
Sphere(<Point>,<Point>)	Sphere Command - The use of Sphere(A,B) will create a sphere with center A and passing through the point B.


B = (8.7, 0.56)
C = (8.6, 9.48)
t1 = Polygon (A, B, C)
State Council of Educational
Research and Training (SCERT)